Extra Examples of Inverse Trigonometric Integrals-Math 121-Benedetto

1.

$$\int \frac{1}{16+x^2}\,dx \qquad \text{not quite in the form of } \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$= \int \frac{1}{16\left(1+\frac{x^2}{16}\right)}\,dx \qquad \text{factor out 16 to get 1 in lead position}$$

$$= \frac{1}{16}\int \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx \qquad \text{rewrite as perfect square}$$

$$= \frac{4}{16}\int \frac{1}{1+w^2}\,dw \qquad \text{substitution} \longrightarrow \text{Here } \boxed{\begin{aligned} w &= \tfrac{x}{4} \\ dw &= \tfrac{1}{4}\,dx \\ 4dw &= dx \end{aligned}}$$

$$= \frac{1}{4}\arctan w + C \qquad \text{antidifferentiate plus constant of integration}$$

$$= \boxed{\frac{1}{4}\arctan\left(\frac{x}{4}\right) + C} \qquad \text{resubstitute back to original variable } x$$

2.

$$\int \frac{1}{\sqrt{25-x^2}}\,dx \qquad \text{not quite in the form of } \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{\sqrt{25\left(1-\frac{x^2}{25}\right)}}\,dx \qquad \text{factor out 25 to get 1 in lead position}$$

$$\int \frac{1}{\sqrt{25}\sqrt{1-\left(\frac{x}{5}\right)^2}}\,dx \qquad \text{rewrite as perfect square}$$

$$= \frac{5}{5}\int \frac{1}{\sqrt{1-w^2}}\,dw \qquad \text{substitution} \longrightarrow \text{Here } \boxed{\begin{aligned} w &= \tfrac{x}{5} \\ dw &= \tfrac{1}{5}\,dx \\ 5dw &= dx \end{aligned}}$$

$$= \arcsin w + C \qquad \text{antidifferentiate plus constant of integration}$$

$$= \boxed{\arcsin\left(\frac{x}{5}\right) + C} \qquad \text{resubstitute back to original variable } x$$

Note: What is the subtle difference between these two integrals? The final answer for number one, has a constant out front of the inverse trigonometric function, with a factor of $\frac{1}{4}$, whereas for number two, there is no constant out front. Think about why there is **no** factor of $\frac{1}{5}$ for number two? In general, using similar proofs (try them?) as above, we have the following antiderivatives:

$$\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C} \qquad \text{with } a \text{ constant}$$

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C} \qquad \text{with } a>0 \text{ constant}$$

You are free to use these two antiderivatives in the future, but I caution you. Do not mix them up, as they are very similar. (I usually think that tangent has an a in the word, so the answer has a $\frac{1}{a}$ factor, but sine does not have an a, so the answer also does not have a $\frac{1}{a}$ factor.) **Plus**, it is always fair game to ask you to show why those statements are true. If you use these facts but misremember them, then you will lose the chance at partial credit, by not showing your work.

For example, practice showing why

Ex: $\displaystyle\int \frac{1}{7+x^2}\,dx = \frac{1}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right)+C$

or

Ex: $\displaystyle\int \frac{1}{\sqrt{3-x^2}}\,dx = \arcsin\left(\frac{x}{\sqrt{3}}\right)+C$

Finally, we need to contrast these examples above with the following (slightly) more basic integrals:

Ex: $\displaystyle\int \frac{1}{1+9x^2}\,dx = \int \frac{1}{1+(3x)^2}\,dx = \frac{1}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{3}\arctan u + C = \boxed{\frac{1}{3}\arctan(3x)+C}$

Here $\boxed{\begin{aligned} u &= 3x \\ du &= 3\,dx \\ \tfrac{1}{3}du &= dx \end{aligned}}$

Ex: $\displaystyle\int \frac{1}{\sqrt{1-49x^2}}\,dx = \int \frac{1}{\sqrt{1-(7x)^2}}\,dx = \frac{1}{7}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{7}\arcsin u + C = \boxed{\frac{1}{7}\arcsin(7x)+C}$

Here $\boxed{\begin{aligned} u &= 7x \\ du &= 7\,dx \\ \tfrac{1}{7}du &= dx \end{aligned}}$

The following examples will help give you some perspective on good choices for substitutions in integration. Lots of training is needed, but so is an understanding on how different integrals relate.

3. $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \boxed{\arctan e^x + C}$

Here $\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}}$ Question: Why is the entire denominator not a good choice for u?

4. $\displaystyle\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln(1+e^x) + C}$

Here $\boxed{\begin{aligned} u &= 1+e^x \\ du &= e^x\,dx \end{aligned}}$ Question: Why is only e^x not an entirely helpful choice for u?

5. $\displaystyle\int \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln\left(1+e^{2x}\right) + C}$

Here $\boxed{\begin{aligned} u &= 1+e^{2x} \\ du &= 2e^{2x}\,dx \\ \tfrac{1}{2}du &= e^{2x}\,dx \end{aligned}}$ Question: Why choose the entire denominator for u?

6. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u + C = \boxed{\arcsin e^x + C}$

Here $\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}}$

7. $\displaystyle\int \frac{e^x}{\sqrt{1-e^x}}\,dx = -\int \frac{1}{\sqrt{u}}\,du = -2\sqrt{u} + C = \boxed{-2\sqrt{1-e^x} + C}$

Here $\boxed{\begin{aligned} u &= 1-e^x \\ du &= -e^x\,dx \\ -du &= e^x\,dx \end{aligned}}$

8. $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{2x}}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}(2)\sqrt{u} + C = \boxed{-\sqrt{1-e^{2x}} + C}$

Here $\boxed{\begin{aligned} u &= 1-e^{2x} \\ du &= -2e^{2x}\,dx \\ -\tfrac{1}{2}du &= e^{2x}\,dx \end{aligned}}$

Extra Examples:

9. $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}\,dx = \int \frac{1}{\sqrt{x}(1+(\sqrt{x})^2)}\,dx = 2\int \frac{1}{1+w^2}\,dw = 2\arctan w + C = \boxed{2\arctan\sqrt{x}+C}$

Here $\boxed{\begin{array}{rl} w & = \sqrt{x} \\ dw & = \frac{1}{2\sqrt{x}}\,dx \\ 2dw & = \frac{1}{\sqrt{x}}dx \end{array}}$

10. $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx = 2\int \frac{1}{w}\,dw = 2\ln|w| + C = \boxed{2\ln|1+\sqrt{x}|+C}$

Here $\boxed{\begin{array}{rl} w & = 1+\sqrt{x} \\ dw & = \frac{1}{2\sqrt{x}}\,dx \\ 2dw & = \frac{1}{\sqrt{x}}dx \end{array}}$ Question: Why are the answers for 9 and 10 so different?

11. $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}\,dx = \int \frac{x^2}{\sqrt{1-(x^3)^2}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{1-w^2}}\,dw = \frac{1}{3}\arcsin w + C = \boxed{\frac{1}{3}\arcsin(x^3)+C}$

Here $\boxed{\begin{array}{rl} w & = x^3 \\ dw & = 3x^2\,dx \\ \frac{1}{3}dw & = x^2dx \end{array}}$

For each one of these integrals in this packet, I recommend that you take the derivative of your answer, using the Chain Rule, and check if you get back to the original integrand. It's a free double check for yourself, but it also helps build up understanding of why each of the individual "compensating" constants are in the answer.

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We end with a slightly complicated example. It uses a slick technique.

First the slick "slip-in/slip-out" trick (or long division of polynomials, if you prefer):

(**)$\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\,dx = \int 1 - \frac{1}{1+x^2}\,dx = \boxed{x - \arctan x + C}$

Note: there are other ways to do this integral (using Trig. Substitution, to come), but this works for now. Finally,

Ex: $\displaystyle\int \frac{e^{3x}}{1+e^{2x}}\,dx = \int \frac{e^{2x}e^x}{1+e^{2x}}\,dx = \int \frac{(e^x)^2\,e^x}{1+(e^x)^2}\,dx = \int \frac{w^2}{1+w^2}\,dw \overset{(**)}{=} \int \frac{w^2+1-1}{1+w^2}\,dw$

$\displaystyle= \int \frac{w^2+1}{1+w^2} - \frac{1}{1+w^2}\,dw = \int 1 - \frac{1}{1+w^2}\,dw = w - \arctan w + C = \boxed{e^x - \arctan(e^x) + C}$

Here $\boxed{\begin{array}{rl} w & = e^x \\ dw & = e^x\,dx \end{array}}$