Exam 3 Fall 2022 Answer Key

1 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (5x+1)^n}{(5n+1)\cdot 4^n}$ Ratio Test

Converges by Ratio Test when

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(5x+1)^{n+1}}{(5(n+1)+1)\cdot 4^{n+1}}}{\dfrac{(-1)^n(5x+1)^n}{(5n+1)\quad 4^n}} \right| = \lim_{n\to\infty} \underbrace{\left|\frac{(5x+1)^{n+1}}{(5x+1)^n}\right|}_{|5x+1|} \cdot \underbrace{\left(\frac{5n+1}{5n+6}\right)}_{1} \cdot \frac{\cancel{4^n}}{\underbrace{4^{n+1}}_{\cancel{4^n}\cdot 4}} = \frac{|5x+1|}{4} < 1$$

$$\frac{|5x+1|}{4} < 1 \Rightarrow |5x+1| < 4 \Rightarrow -4 < 5x+1 < 4 \Rightarrow -5 < 5x < 3 \Rightarrow -1 < x < \frac{3}{5}$$

$\quad -1 \quad -1 \quad -1$

Manually Check Convergence at Endpoints

Take $x=-1$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n[5(-1)+1]^n}{(5n+1)\cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-4)^n}{(5n+1)\cdot 4^n} = \sum_{n=1}^{\infty} \frac{\overbrace{\cancel{(-1)^n(-1)^n}}^{(-1)^{2n} \to 1 \text{ Even}}\cancel{4^n}}{(5n+1)\cdot \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{1}{5n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges (Harmonic) p-Series p = 1

$$\lim_{n\to\infty} \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} \to \frac{1}{5} = \frac{1}{5} \quad \text{Finite, Non-Zero}$$

$\Rightarrow$ Series also Diverges by LCT

Take $x = \frac{3}{5}$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n[\cancel{5}(\frac{3}{\cancel{5}})+1]^n}{(5n+1)\ 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{4^n}}{(5n+1)\cancel{4^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$$

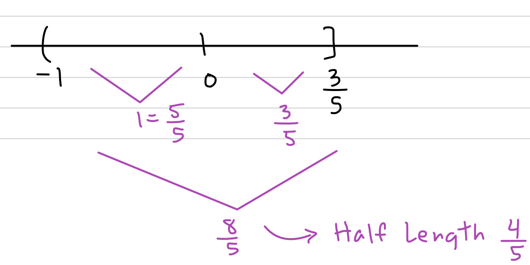
1. Isolate $b_n = \frac{1}{5n+1} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5n+1} \to \frac{1}{\infty} \to 0 = 0$
3. Terms Decreasing

$$b_{n+1} = \frac{1}{5(n+1)+1} = \frac{1}{5n+6} \le \frac{1}{5n+1} = b_n$$

Series Converges by AST

Finally, Interval of Convergence $I = \left(-1, \frac{3}{5}\right]$

Radius of Convergence $R = \frac{4}{5}$



2a. $\dfrac{x^3}{7+x} = x^3\left(\dfrac{1}{7+x}\right) = \dfrac{x^3}{7}\left(\dfrac{1}{1+\frac{x}{7}}\right) = \dfrac{x^3}{7}\left(\dfrac{1}{1-\left(-\frac{x}{7}\right)}\right) = \dfrac{x^3}{7}\sum_{n=0}^{\infty}\left(-\dfrac{x}{7}\right)^n$

$$= \frac{x^3}{7}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+3}}{7^{n+1}}$$

Need $\left|-\dfrac{x}{7}\right| = \dfrac{|x|}{7} < 1$

$|x| < 7$

$-7 < x < 7$

$R = 7$

2b. $6x^3 \arctan(6x) = 6x^3 \sum_{n=0}^{\infty}\dfrac{(-1)^n (6x)^{2n+1}}{2n+1} = 6x^3\sum_{n=0}^{\infty}\dfrac{(-1)^n 6^{2n+1} x^{2n+1}}{2n+1}$

Need $|6x| < 1$

$|x| < \frac{1}{6}$

$-\frac{1}{6} < x < \frac{1}{6}$

$R = \frac{1}{6}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n 6^{2n+2} x^{2n+4}}{2n+1}$$

3a. $\dfrac{d}{dx}\left(8x^4 \ln(1+8x)\right) = \dfrac{d}{dx}\, 8x^4 \sum_{n=0}^{\infty}\dfrac{(-1)^n (8x)^{n+1}}{n+1} = \dfrac{d}{dx}\, 8x^4\sum_{n=0}^{\infty}\dfrac{(-1)^n 8^{n+1} x^{n+1}}{n+1}$

$$= \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n 8^{n+2} x^{n+5}}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n 8^{n+2}(n+5)x^{n+4}}{n+1}$$

3b. $\displaystyle\int x^2 e^{-x^3}\,dx = \int x^2 \sum_{n=0}^{\infty}\frac{(-x^3)^n}{n!}\,dx = \int x^2\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!}\,dx$

$$= \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+2}}{n!}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+3}}{n!\,(3n+3)} + C$$

4. $$\int_0^1 x^2\cos(x^3)\,dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}\,dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!\,(6n+3)} \Bigg|_0^1$$

$n=0 \quad n=1 \quad n=2$

$$= \frac{x^3}{1\cdot 3} - \frac{x^9}{2!\cdot 9} + \frac{x^{15}}{\underset{24}{4!}\cdot(15)} - \dots \Bigg|_0^1$$

$$= \frac{1}{3} - \frac{1}{18} + \frac{1}{360} - \dots - \cancel{(0 - 0 + 0 - \dots)}^{\,0}$$

$$\approx \frac{1}{3} - \frac{1}{18} = \frac{6}{18} - \frac{1}{18} = \frac{5}{18} \quad \longleftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem (A.S.E.T)
we can Estimate the Full Sum using only the
first two terms with error at most $\frac{1}{360} < \frac{1}{50}$ as desired

5a. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n} \cdot \frac{\pi}{3}}{(2n+1)! \cdot \frac{\pi}{3}}$$

$9^n = (3^2)^n = 3^{2n}$

$$= \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{3}{\pi} \cancel{\sin\left(\frac{\pi}{3}\right)}^{\frac{\sqrt{3}}{2}} = \frac{3\sqrt{3}}{2\pi}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

5b. $$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right)$$

$$= -\ln(1+1) = -\ln 2$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

5c. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n+1}}{4!\,(2n)!} = -\frac{\pi}{24}\sum_{n=0}^{\infty}\frac{(-1)^n \pi^{2n}}{(2n)!} = -\frac{\pi}{24}\cdot \overset{-1}{\cancel{\cos\pi}} = \frac{\pi}{24}$

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

5d $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}(\ln 3)^n}{5\cdot n!} = \frac{2}{5}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n (\ln 3)^n}{n!} = \frac{2}{5}\sum_{n=0}^{\infty}\frac{(-2\ln 3)^n}{n!}$

$= \frac{2}{5} e^{-2\ln 3} = \frac{2}{5}\cancel{e}^{\cancel{\ln}(3^{-2})} = \frac{2}{5}\cdot 3^{-2} = \frac{2}{5}\cdot\frac{1}{9} = \frac{2}{45}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

5e. $-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\ldots = (4\,\overset{\frac{\pi}{4}}{\cancel{\arctan 1}}) - 4 = \cancel{4}\cdot\frac{\pi}{\cancel{4}} - 4 = \pi - 4$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \ldots$

$4\arctan 1 = 4\left(1-\frac{1}{3}+\frac{1}{5}-\ldots\right) = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \ldots$

5f $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n} = \sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n = \frac{1}{1-\left(-\frac{2}{3}\right)} = \frac{1}{\frac{5}{3}} = \frac{3}{5}$

$+$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

6. $$\lim_{x\to 0}\frac{1-\cos(2x)}{3x^2+x-\arctan x} = \lim_{x\to 0}\frac{1-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\right)}{3x^2+x-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots\right)}$$

$$= \lim_{x\to 0}\frac{\cancel{1}-\cancel{1}+\frac{2^2x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}-\dots}{3x^2+\cancel{x}-\cancel{x}+\frac{x^3}{3}-\frac{x^5}{5}+\dots}$$ Need all ...

$$= \lim_{x\to 0}\frac{\frac{2^2x^2}{2!}-\frac{2^4x^4}{4!}+\frac{2^6x^6}{6!}-\dots}{3x^2+\frac{x^3}{3}-\frac{x^5}{5}+\dots} \quad \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0}\frac{\overset{2}{\frac{4}{2}}\cancelto{0}{-\frac{2^4x^2}{4!}+\frac{2^6x^4}{6!}}-\dots}{3\cancelto{0}{+\frac{x}{3}-\frac{x^3}{5}}+\dots} = \frac{2}{3}$$

Not Required this time, but can check answer with L'Hôpital's Rule

$$\lim_{x\to 0}\frac{1-\cos(2x)}{3x^2+x-\arctan x}\ \frac{0}{0} \underset{L'H}{=} \lim_{x\to 0}\frac{2\sin(2x)}{6x+1-\frac{1}{1+x^2}}\ \frac{0}{0} \underset{L'H}{=} \lim_{x\to 0}\frac{4\cancelto{1}{\cos(2x)}}{6+\cancelto{0}{\frac{2x}{(1+x^2)^2}}} = \frac{4}{6} = \frac{2}{3}$$ Match!

7. $$\arctan x = \int\frac{1}{1+x^2}\,dx = \int\frac{1}{1-(-x^2)}\,dx = \int\sum_{n=0}^{\infty}(-x^2)^n\,dx = \int\sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} + C$$ Solve for C

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \cancelto{0}{C}$$

Test $x=0$

$$\cancelto{0}{\arctan 0} = 0 - \cancelto{0}{0+0} - \dots + C \Rightarrow 0 = 0 + C \Rightarrow C = 0$$

Finally, $$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

Optional Bonus #1

$\sum_{n=0}^{\infty} \frac{n}{3^n}$ looks like $\sum_{n=0}^{\infty} n x^n$ where $x = \frac{1}{3}$

work in reverse to find function

$$\sum_{n=0}^{\infty} n x^n = \sum_{n=0}^{\infty} n \cdot x^{n-1} \cdot x = x \sum_{n=0}^{\infty} n x^{n-1} = x \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$\frac{1}{1-x}$

looks like derivative

$$= x \frac{d}{dx}\left(\frac{1}{1-x}\right) = x\left(\frac{1}{(1-x)^2}\right) = \frac{x}{(1-x)^2}$$

$(1-x)^{-1} = -(1-x)^{-1} \cdot (-1) = \overset{+}{(1-x)}^{-2}$

Plug in $x = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{n}{3^n} = \sum_{n=0}^{\infty} n\left(\frac{1}{3}\right)^n = \frac{\frac{1}{3}}{\left(1-\frac{1}{3}\right)^2} = \frac{\frac{1}{3}}{\frac{4}{9}} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{9}^{\,3}}{4} = \frac{3}{4}$$

$\frac{2}{3}$ $\quad \frac{9}{4}$

Optional Bonus #2

$$x^3 \arctan(x^5) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (x^5)^{2n+1}}{2n+1} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+5}}{2n+1}$$

n = 0 $\quad$ n = 1 $\quad$ n = 2 $\quad$ n = 3

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{10n+8}}{2n+1} = \frac{x^8}{1} - \frac{x^{18}}{3} + \frac{x^{28}}{5} - \frac{x^{38}}{7} + \cdots$$

$0x^{27}$

coefficient 0

Match to General MacLaurin Series

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(27)}(0)}{(27)!}x^{27} + \frac{f^{(28)}(0)}{(28)!}x^{28} + \cdots$$

Equate coefficients of like Powered Terms

$$\frac{f^{(27)}(0)}{(27)!} = 0 \Rightarrow f^{(27)}(0) = 0$$

$$\frac{f^{(28)}(0)}{(28)!} = \frac{1}{5} \Rightarrow f^{(28)}(0) = \frac{(28)!}{5}$$