Exam 2 Fall 2022 Answer Key

1(a) $\displaystyle\int_0^e x\ln x\,dx = \lim_{t\to 0^+}\int_t^e x\ln x\,dx = \lim_{t\to 0^+} \frac{x^2}{2}\ln x\Big|_t^e - \frac{1}{2}\int_t^e x\,dx$

IBP

$\boxed{\begin{array}{ll} u=\ln x & dv = x\,dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2}\end{array}}$

$$= \lim_{t\to 0^+} \frac{x^2}{2}\ln x\Big|_t^e - \frac{x^2}{4}\Big|_t^e$$

$$= \lim_{t\to 0^+} \frac{e^2}{2}\cdot\cancelto{1}{\ln e} - \cancelto{0}{\frac{t^2}{2}\ln t} - \left(\frac{e^2}{4} - \cancelto{0}{\frac{t^2}{4}}\right) \quad \star$$

$$= \frac{e^2}{2} - \frac{e^2}{4} = \frac{e^2}{4}$$

Match! Converges

$\star$ $\displaystyle\lim_{t\to 0^+} t^2\ln t \; (0\cdot(-\infty)) = \lim_{t\to 0^+}\frac{\ln t}{1/t^2}\; \frac{-\infty}{\infty} \underset{L'H}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{2}{t^3}} \; \left(-\frac{t^3}{2}\right) = \lim_{t\to 0^+}\frac{\cancelto{0}{-t^2}}{2} = 0$

1(b) $\displaystyle\int_0^{1/2}\frac{1}{x\ln x}dx = \lim_{t\to 0^+}\int_t^{1/2}\frac{1}{x\ln x}dx = \lim_{t\to 0^+}\int_{\ln t}^{\ln 1/2}\frac{1}{u}du = \lim_{t\to 0^+}\ln|u|\Big|_{\ln t}^{\ln 1/2}$

$\boxed{\begin{array}{l} u=\ln x \\ du = \frac{1}{x}dx\end{array}}$

$$= \lim_{t\to 0^+} \underbrace{\ln|\ln 1/2|}_{\text{Finite}} - \cancelto{\infty}{\ln\left|\cancelto{|-\infty|}{\ln \cancelto{0^+}{t}}\right|}$$

$\boxed{\begin{array}{l} x=t \Rightarrow u=\ln t \\ x=1/2 \Rightarrow u = \ln 1/2\end{array}}$

$$= -\infty$$

Diverges Match!

1(c) $\displaystyle\int_3^{\infty}\frac{20-x}{x^2-4x+7}dx = \lim_{t\to\infty}\int_3^t\frac{20-x}{x^2-4x+7}dx = \lim_{t\to\infty}\int_3^t\frac{20-x}{(x-2)^2+3}dx$

Discriminant:

$b^2-4ac = 16-4(1)(7) = -12 < 0$

↳ Complete the Square

inversion

$\boxed{\begin{array}{l} u = x-2 \Rightarrow x = u+2 \\ du = dx\end{array}}$ $\displaystyle = \lim_{t\to\infty}\int_1^{t-2}\frac{20-(u+2)}{u^2+3}du$

$\boxed{\begin{array}{l} x=3 \Rightarrow u = 3-2 = 1 \\ x = t \Rightarrow u = t-2\end{array}}$ $\displaystyle = \lim_{t\to\infty}\int_1^{t-2}\frac{18-u}{u^2+3}du$ split-split

$$= \lim_{t\to\infty}\frac{18}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right) - \frac{1}{2}\ln|u^2+3|\Big|_1^{t-2}$$

$$= \lim_{t\to\infty}\underbrace{\frac{18}{\sqrt{3}}\cancelto{\frac{\pi}{2}}{\arctan\left(\frac{\cancelto{\infty}{t-2}}{\sqrt{3}}\right)}}_{\text{Finite}} \cancelto{\infty}{- \frac{1}{2}\ln\left|\cancelto{\infty}{(t-2)^2+3}\right|} - \left(\underbrace{\frac{18}{\sqrt{3}}\cancelto{\frac{\pi}{6}}{\arctan\left(\frac{1}{\sqrt{3}}\right)}}_{\text{Finite}} - \underbrace{\frac{1}{2}\ln 4}_{\text{Finite}}\right) = -\infty$$

Match! Diverges

$$1(d)\quad \int_{-4}^{3} \frac{20-x}{x^2-4x-32}\,dx = \int_{-4}^{3} \frac{20-x}{(x-8)(x+4)}\,dx = \lim_{t\to -4^+} \int_t^3 \frac{20-x}{(x-8)(x+4)}\,dx$$

V.A.

Partial Fractions Decomposition

$$\cancel{(x-8)(x+4)}\left(\frac{20-x}{(x-8)\cancel{(x+4)}} = \frac{A}{x-8} + \frac{B}{x+4}\right)(x-8)(x+4)$$

$$20-x = A(x+4)+B(x-8)$$
$$= Ax+4A+Bx-8B$$
$$=(A+B)x+(4A-8B)$$

Conditions:

- $A+B=-1 \Rightarrow B=-1-A$
- $4A-8B=20$

$$4A-8(-1-A)=20$$
$$4A+8+8A=20$$
$$12A=12$$
$$A=1 \Rightarrow B=-2$$

PFD

$$= \lim_{t\to -4^+} \int_t^3 \frac{1}{x-8} - \frac{2}{x+4}\,dx$$

$$= \lim_{t\to -4^+} \ln|x-8| - 2\ln|x+4| \Big|_t^3$$

$$= \lim_{t\to -4^+} \underbrace{\ln|-5| - 2\ln 7 - \Big(\overset{\ln(12)}{\ln|t-8|}}_{\text{Finite}} \overset{-\infty}{-} 2\ln|\overset{0^+}{t+4}|\Big) \quad t\to -4^+$$

ln 5

$$= -\infty \quad \text{Match! Diverges}$$

Warning: $\ln 0$ is undefined, must "sneak attack" using the Improper Limit

$$1(e)\quad \int_0^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+} -\int_{1/t}^{1} e^u\,du = \lim_{t\to 0^+} -e^u \Big|_{1/t}^{1}$$

$$u=\frac{1}{x}$$
$$du = \frac{-1}{x^2}\,dx$$

$$x=t \Rightarrow u=\frac{1}{t}$$
$$x=1 \Rightarrow u=1$$

$$= \lim_{t\to 0^+} -e + \overset{\infty}{e^{\overset{\infty}{1/t}}} \quad t\to 0^+$$

$$= \infty \quad \text{Match! Diverges}$$

$$1(f)\quad \int_1^{\infty} \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{e^{1/x}}{x^2}\,dx = \dots = \lim_{t\to\infty} -e^{1/x}\Big|_1^t$$

reuse u-sub above

$$= \lim_{t\to\infty} \boxed{-}\overset{1}{e^{\overset{0}{1/t}}} + e = e-1 \quad \text{Match! Converges}$$

$t \to \infty$

1(g) $\displaystyle\int_0^{e^3} \frac{1}{x[9+(\ln x)^2]}\,dx = \lim_{t\to 0^+} \int_t^{e^3} \frac{1}{x[9+(\ln x)^2]}\,dx = \lim_{t\to 0^+} \int_{\ln t}^{3} \frac{1}{9+u^2}\,du$

$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$

$$= \lim_{t\to 0^+} \frac{1}{3} \arctan\left(\frac{u}{3}\right)\Bigg|_{\ln t}^{3}$$

$\boxed{\begin{aligned} x = t &\Rightarrow u = \ln t \\ x = e^3 &\Rightarrow u = \cancel{\ln} e^3 = 3 \end{aligned}}$

$$= \lim_{t\to 0^+} \frac{1}{3}\left(\arctan\left(\frac{3}{3}\right) - \arctan\left(\frac{\ln t}{3}\right)\right)$$

$\frac{3}{3}\to 1$, $\arctan(1)\to\frac{\pi}{4}$; $t\to 0^+$, $\ln t\to -\infty$, $\arctan\left(\frac{\ln t}{3}\right)\to -\frac{\pi}{2}$

$$= \frac{1}{3}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{1}{\cancel{3}}\left(\frac{\cancel{3}\pi}{4}\right) = \frac{\pi}{4}$$ Match! Converges

2. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{\ln n}$ 1st Diverges by nTDT because

$$\lim_{n\to\infty} \frac{e^n}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{e^x}{\ln x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{e^x}{1/x} \overset{x}{=} \lim_{x\to\infty} x e^x = \infty \neq 0$$

2nd Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{e^{n+1}}{\ln(n+1)}}{\frac{e^n}{\ln n}}\right| \overset{\frac{\ln n}{e^n}}{=} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \cdot \frac{e^{n+1}}{e^n} = e > 1$$

$\frac{\ln n}{\ln(n+1)} \to 1$ (☆), $\frac{e^{n+1}}{e^n} = e$

Diverges by Ratio Test

☆ $$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} \overset{x+1}{=} \lim_{x\to\infty} \frac{x+1}{x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\cos^2 n}{n^6+7} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^6+7} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$ Converges p-Series $p=6>1$

Bound Terms:

$$\frac{\cos^2 n}{n^6+7} \leq \frac{1}{n^6+7} \leq \frac{1}{n^6}$$

$\Rightarrow$ Absolute Series Converges by CT

$\xLeftarrow{ACT}$

Original Series Converges by ACT

4(a) $\sum_{n=1}^{\infty} \frac{n^6+7}{7n^6+6}$ Diverges by nTDT because

$$\lim_{n\to\infty} \frac{n^6+7}{7n^6+6} \cdot \frac{\frac{1}{n^6}}{\frac{1}{n^6}} = \lim_{n\to\infty} \frac{1+\frac{7}{n^6}}{7+\frac{6}{n^6}} = \frac{1}{7} \neq 0$$

4(b) $\sum_{n=1}^{\infty} \frac{6n!}{7n^n}$ Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{6(n+1)!}{7(n+1)^{n+1}}}{\frac{6n!}{7n^n}} \right| = \lim_{n\to\infty} \frac{6}{6} \cdot \frac{7}{7} \cdot \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \frac{1}{e} < 1$$

$(n+1)!=(n+1)\,n!$, $(n+1)^{n+1}=(n+1)^n(n+1)$

Series Absolutely Converges by Ratio Test

Since the Series is already the Absolute Series, then Absolute Convergence is the same as Convergence

(OR, Make ACT Argument, but not needed since OS=AS)

4(c) $\sum_{n=1}^{\infty} \frac{n^6+7}{n^7} \approx \sum_{n=1}^{\infty} \frac{n^6}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges (Harmonic) p-Series $p=1$

Bound Terms

$$\frac{n^6+7}{n^7} \geq \frac{n^6}{n^7} = \frac{1}{n}$$

$\Rightarrow$ Original Series also Diverges by CT

Note: LCT Limit also Works.

4(d) $\sum_{n=1}^{\infty} \frac{1}{n^7+6} + \frac{6^n}{7^n} = \sum_{n=1}^{\infty} \frac{1}{n^7+6} + \sum_{n=1}^{\infty} \frac{6^n}{7^n}$

$\sum_{n=1}^{\infty} \frac{1}{n^7+6} \approx \sum_{n=1}^{\infty} \frac{1}{n^7}$ Converges p-Series $p=7>1$

Converges by GST b/c $|r| = \frac{6}{7} < 1$

Bound Terms

$\frac{1}{n^7+6} \leq \frac{1}{n^7}$ $\Rightarrow$ Left Series Converges by CT

$+$

Original Series Converges b/c Sum of 2 Convergent Series is Convergent

4(e) $\displaystyle\sum_{n=1}^{\infty}\left(1-\frac{7}{n^6}\right)^{n^6}$ Diverges by nTDT because

1^{∞}

$$\lim_{n\to\infty}\left(1-\frac{7}{n^6}\right)^{n^6}=\lim_{x\to\infty}\left(1-\frac{7}{x^6}\right)^{x^6}=e^{\lim_{x\to\infty}\ln\left(\left(1-\frac{7}{x^6}\right)^{x^6}\right)}=e^{\lim_{x\to\infty}x^6\ln\left(1-\frac{7}{x^6}\right)}$$

$$=e^{\lim_{x\to\infty}\frac{\ln\left(1-\frac{7}{x^6}\right)}{\frac{1}{x^6}}}\overset{0/0}{\underset{L'H}{=}}e^{\lim_{x\to\infty}\frac{\frac{1}{1-\frac{7}{x^6}}\cdot\left(\frac{42}{x^7}\right)}{\frac{-6}{x^7}}}=e^{-7}\neq 0$$

5. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx=\lim_{t\to\infty}\int_2^t\frac{1}{x(\ln x)^2}dx=\lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{1}{u^2}du=\lim_{t\to\infty}-\frac{1}{u}\Big|_{\ln 2}^{\ln t}$$

$\boxed{u=\ln x \quad du=\frac{1}{x}dx}$ $\boxed{x=2\Rightarrow u=\ln 2 \quad x=t\Rightarrow u=\ln t}$

$$=\lim_{t\to\infty}-\frac{1}{\ln t}+\frac{1}{\ln 2}=\frac{1}{\ln 2}$$ Integral Converges

$\Rightarrow$ Series Converges by Integral Test

6(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{7n+6}\xrightarrow{A.S.}\sum_{n=1}^{\infty}\frac{1}{7n+6}\approx\sum_{n=1}^{\infty}\frac{1}{n}$ Diverges Harmonic p-Series $p=1$

$$\lim_{n\to\infty}\frac{\frac{1}{7n+6}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{7n+6}\cdot\frac{1/n}{1/n}=\lim_{n\to\infty}\frac{1}{7+\frac{6}{n}}=\frac{1}{7}$$ Finite Non-Zero

$\Rightarrow$ Absolute Series Diverges by LCT

AST

1 Isolate $b_n=\frac{1}{7n+6}>0$

2. $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{7n+6}=0$

3. Terms Decreasing

$$b_{n+1}=\frac{1}{\underset{7n+13}{7(n+1)+6}}\leq\frac{1}{7n+6}=b_n$$

$\Rightarrow$ Original Series Converges by AST

Original Series Conditionally Convergent by Definition

6(b) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^n \cdot (2n)!}{n^6 \cdot 6^n \cdot (n!)^3}$ Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} (n+1)^{n+1} (2(n+1))!}{(n+1)^6 \cdot 6^{n+1} ((n+1)!)^3}}{\frac{\cancel{(-1)^n} \cdot n^n (2n)!}{n^6 \cdot 6^n (n!)^3}} \right|$$

$$(n+1)^n (n+1) \quad (2n+2)(2n+1)\cancel{(2n)!} \quad \left(\frac{n \frac{1}{n}}{n+1 \frac{1}{n}}\right)^6$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{(2n+2)!}{\cancel{(2n)!}} \cdot \frac{n^6}{(n+1)^6} \cdot \frac{\cancel{6^n}}{6^{n+1}} \cdot \frac{\cancel{(n!)^3}}{((n+1)!)^3}$$

$$\cancel{6^n} \cdot 6 \qquad ((n+1)n!)^3 \qquad (n+1)^3 \cancel{(n!)^3}$$

$$= \lim_{n\to\infty} \frac{1}{6} \cdot \frac{(n+1)^n}{n^n}^{\to e} \cdot \frac{\cancel{n+1}}{\cancel{n+1}} \cdot \frac{\overbrace{2n+2}^{2\cancel{(n+1)}}}{\cancel{n+1}} \cdot \frac{2n+1 \frac{1}{n}}{n+1 \frac{1}{n}} \cdot \left(\frac{1}{1+\frac{1}{n}\to 0}\right)^{6} \to 1$$

$$= \lim_{n\to\infty} \frac{2e}{6} \cdot \left(\frac{2+\frac{1}{n}\to 0}{1+\frac{1}{n}\to 0}\right)\to 2 = \frac{4e}{6} = \frac{2e}{3} > 1$$ Series Diverges by Ratio Test

$e \approx 2.718$

$\hookrightarrow 2e > 4 > 3$

6(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+6n+7}{n^6+7n+6} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{n^2+6n+7}{n^6+7n+6} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ Converges p-Series

$p = 4 > 1$

$$\lim_{n\to\infty} \frac{\frac{n^2+6n+7}{n^6+7n+6} \; n^4}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^6+6n^5+7n^4 \; \frac{1}{n^6}}{n^6+7n+6 \; \frac{1}{n^6}} = \lim_{n\to\infty} \frac{1+\frac{6}{n}\to 0+\frac{7}{n^2}\to 0}{1+\frac{7}{n^5}\to 0+\frac{6}{n^6}\to 0} = 1$$ Finite Non-Zero

$\Rightarrow$ Absolute Series also Converges by LCT

$\Rightarrow$ original Series Absolutely Convergent by Definition