Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121

Midterm Exam #2

February 15-18, 2020

• This is an *Open Notes* Exam. You can use materials, homeworks problems, lecture notes, etc. that you manually worked on.

• There is **NO** *Open Internet* allowed. You can only access our Main Course Webpage.

• You are not allowed to work on or discuss these problems with other people. You can ask a few small, clarifying, questions in Office Hours, but these problems will not be solved.

• Submit your final work in Gradescope in the Exam 2 entry.

• Please *show* all of your work and *justify* all of your answers.

Problem	Score	Possible Points
1		40
2		14
3		21
4		25
Total		100

1. [40 Points] Compute the following integrals. Justify your work.

(a) $\displaystyle\int \frac{11-x}{x^2-4x+5}\,dx$ $\overset{\text{Complete Square}}{=} \displaystyle\int \frac{11-x}{(x-2)^2+1}\,dx = \int \frac{11-(u+2)}{u^2+1}\,du$ $\quad 11-u-2$

$b^2-4ac = \underbrace{16-4(1)(5)}_{-4} < 0$

$(x-2)^2 = x^2-4x+4$ $\quad +1$

$\boxed{u = x-2 \Rightarrow x = u+2 \quad du = dx}$

$$= \int \frac{9-u}{u^2+1}\,du$$

$$= \int \frac{9}{u^2+1} - \frac{u}{u^2+1}\,du$$

$$= 9\arctan u - \frac{1}{2}\ln|u^2+1| + C$$

$$= \boxed{9\arctan(x-2) - \frac{1}{2}\ln\left|\underbrace{(x-2)^2+1}_{x^2-4x+5}\right| + C}$$

(b) $\displaystyle\int_{-2}^{-1} \frac{11-x}{x^2-4x-5}\,dx = \int_{-2}^{-1} \frac{11-x}{(x-5)(x+1)}\,dx = \lim_{t\to -1^-} \int_{-2}^{t} \frac{11-x}{(x-5)(x+1)}\,dx$

Improper

Factor + PFD

PFD

$(x-5)(x+1)\left[\frac{11-x}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}\right](x-5)(x+1)$

$11-x = A(x+1)+B(x-5)$

$= Ax+A+Bx-5B$

$= (A+B)x + A-5B$

Conditions

- $A+B=-1 \quad B=-1-A$
- $A-5B=11$

$A-5(-1-A)=11$

$A+5+5A=11$

$6A=6$

$A=1$

$\Rightarrow B=-1-1=-2$

$$= \lim_{t\to -1^-} \int_{-2}^{t} \frac{1}{x-5} - \frac{2}{x+1}\,dx$$

$$= \lim_{t\to -1^-} \ln|x-5| - 2\ln|x+1| \Big|_{-2}^{t}$$

$$= \lim_{t\to -1^-} \underbrace{\ln|t-5|}_{\ln 6 \text{ Finite}} - \underbrace{2\ln\underbrace{|t+1|}_{|0^-|\to 0^+}}_{-\infty} - \underbrace{[\ln 7 - 2\ln 1]}_{\text{Finite}}$$

$$= \boxed{\infty} \qquad \text{Diverges}$$

1. (Continued) Compute the following integrals. Justify your work. Complete Square

(c) $\displaystyle\int_{-\infty}^{7} \frac{1}{x^2-4x+29}\,dx = \lim_{t\to-\infty}\int_t^7 \frac{1}{x^2-4x+29}\,dx = \lim_{t\to-\infty}\int_t^7 \frac{1}{(x-2)^2+25}\,dx$

$(x-2)^2 = x^2-4x+4$ $+25$

$$= \lim_{t\to-\infty}\int_{t-2}^{5} \frac{1}{u^2+25}\,dx = \lim_{t\to-\infty} \frac{1}{5}\arctan\left(\frac{u}{5}\right)\Big|_{t-2}^{5}$$

$\boxed{\begin{array}{l} u = x-2 \\ du = dx \end{array}}$ $\boxed{\begin{array}{l} x = t \Rightarrow u = t-2 \\ x = 7 \Rightarrow u = 7-2 = 5 \end{array}}$

$$= \lim_{t\to-\infty} \frac{1}{5}\left[\arctan\left(\frac{5}{5}\right) - \arctan\left(\frac{t-2}{5}\right)\right]$$

$\frac{5}{5} \to 1$, $\arctan \to \frac{\pi}{4}$; $\frac{t-2}{5} \to -\infty$, $\arctan \to -\frac{\pi}{2}$

$$= \lim_{t\to-\infty} \frac{1}{5}\left[\frac{\pi}{4}+\frac{\pi}{2}\right] = \frac{1}{5}\left[\frac{3\pi}{4}\right] = \boxed{\frac{3\pi}{20}}$$ Converges

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(d) $\displaystyle\int_0^1 x^5 \ln x\,dx = \lim_{t\to0^+}\int_t^1 x^5 \cdot \ln x\,dx \overset{IBP}{=} \lim_{t\to0^+} \frac{x^6}{6}\ln x\Big|_t^1 - \frac{1}{6}\int_t^1 x^5\,dx$

IBP

$\boxed{\begin{array}{ll} u = \ln x & dv = x^5 dx \\ du = \frac{1}{x}dx & v = \frac{x^6}{6} \end{array}}$

$$= \lim_{t\to0^+} \frac{x^6}{6}\ln x\Big|_t^1 - \frac{x^6}{36}\Big|_t^1$$

$$= \lim_{t\to0^+} \frac{1}{6}\ln 1 - \underbrace{\frac{t^6}{6}\ln t}_{(*)} - \frac{1}{36} + \frac{t^6}{36} = \boxed{-\frac{1}{36}}$$ Converges

($\frac{1}{6}\ln 1 \to 0$, $\frac{t^6}{6}\ln t \to 0$, $\frac{t^6}{36} \to 0$)

$(*)$ $\displaystyle\lim_{t\to0^+} t^6 \cdot \ln t = \lim_{t\to0^+} \frac{\ln t}{1/t^6} \overset{L'H}{=} \lim_{t\to0^+} \frac{1/t}{-6/t^7} = \lim_{t\to0^+} \frac{-t^6}{6} = 0$

$0\cdot(-\infty)$; $\frac{-\infty}{\infty}$; $1/t^6 \to t^{-6} \to -6t^{-7}$; $-\frac{t^7}{6}$; $\to 0$

2. [14 Points] Demonstrate **two different** methods to prove this given series $\sum_{n=1}^{\infty} \frac{n}{e^{2n}}$ Converges.

1. First you must use the Integral Test.
2. Second, use a different method.

1. Related Function $f(x)=\frac{x}{e^{2x}}$

- Positive $x>0$
- Continuous for all x
- Decreasing $f'(x)=\frac{e^{2x}(1)-x\cdot 2e^{2x}}{(e^{2x})^2}$

$$=\frac{\cancel{e^{2x}}(1-2x)}{(e^{2x})^{\cancel{2}}}$$

$$=\frac{1-2x}{e^{2x}}<0 \text{ when } 1-2x<0$$

$\Rightarrow 2x>1$

$\Rightarrow x>\frac{1}{2}$ o.k.

Compute

$$\int_1^\infty \frac{x}{e^{2x}}dx=\lim_{t\to\infty}\int_1^t xe^{-2x}dx=\lim_{t\to\infty}\frac{-x}{2e^{2x}}\Big|_1^t+\frac{1}{2}\int_1^t e^{-2x}dx$$

IBP

$$\boxed{\begin{array}{ll} u=x & dv=e^{-2x}dx \\ du=dx & v=\frac{e^{-2x}}{-2}\end{array}}$$

$\frac{e^{-2x}}{-4}$

$$=\lim_{t\to\infty}\frac{-x}{2e^{2x}}\Big|_1^t-\frac{1}{4e^{2x}}\Big|_1^t$$

$$=\lim_{t\to\infty}\frac{-t}{2e^{2t}}^{-\infty/\infty}+\frac{1}{2e^2}-\cancel{\frac{1}{4e^{2t}}}^{0}+\frac{1}{4e^2}$$

$$\underset{L'H}{=}\lim_{t\to\infty}\cancel{\frac{-1}{4e^{2t}}}^{0}+\frac{1}{2e^2}+\frac{1}{4e^2}=\frac{3}{4e^2}$$ Integral Converges

$\Rightarrow$ O.S. Converges by I.T.

2. Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{n+1}{e^{2(n+1)}}}{\frac{n}{e^{2n}}}\right|=\lim_{n\to\infty}\left(\frac{n+1}{n}\frac{\frac{1}{n}}{\frac{1}{n}}\right)\cdot\frac{\cancel{e^{2n}}}{\cancel{e^{2n}}e^{2}}$$

$$=\lim_{n\to\infty}\left(\frac{1+\cancel{1/n}^{0}}{1}\right)\cdot\frac{1}{e^2}=\frac{1}{e^2}<1$$ $\Rightarrow$ O.S. (Absolutely) Convergent by Ratio Test.

OR O.S. = A.S. So A.C. Same as Converge here

3. [21 Points] Determine whether each of the following series **converges** or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} \frac{7}{n^9} + \frac{7^n}{9^n}$ $= 7\sum_{n=1}^{\infty}\frac{1}{n^9} + \sum_{n=1}^{\infty}\frac{7^n}{9^n}$

split

Constant Multiple of Convergent p-Series $p=9>1$ is Convergent

Convergent Geometric Series (or by GST) with $|r| = |7/9| = 7/9 < 1$

Sum of 2 Convergent Series is Convergent

(b) $\sum_{n=2}^{\infty} \frac{n^9}{7\ln n}$ Divergent by nTDT because

$$\lim_{n\to\infty}\frac{n^9}{7\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{x^9}{7\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{9x^8}{7/x} \cdot \frac{x}{x} = \lim_{x\to\infty}\frac{9x^9}{7} = \infty \neq 0.$$

(c) Use the Absolute Convergence Test to Prove that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^9+7^n}$ is convergent.

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^9+7^n} \xrightarrow{A.S.} \sum_{n=1}^{\infty}\frac{1}{n^9+7^n} \approx \sum_{n=1}^{\infty}\frac{1}{7^n}$$

Convergent GST $|r| = \frac{1}{7} < 1$

Bound Terms

$$\frac{1}{n^9+7^n} \le \frac{1}{7^n}$$ and

$\Rightarrow$ A.S. Converges by C.T.

O.S. Converges by ACT

4. [25 Points] Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{9n-7}$ $\xrightarrow[\text{must}]{\text{A.S.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9n-7} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Divergent Harmonic p-series $p=1$

AST on O.S.

1. $b_n = \dfrac{1}{9n-7} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{9n-7} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underset{\substack{9n+9-7 \\ 9n+2}}{9(n+1)-7}} \le \frac{1}{9n-7} = b_n$$

OR

$$f(x) = \frac{1}{9x-7} = (9x-7)^{-1}$$

$$\Rightarrow f'(x) = \frac{-9}{(9x-7)^2} < 0$$

O.S. Converges by AST

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{9n-7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{9n-7}\cdot\frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{9-7/n} = \frac{1}{9}$$

Finite, Non-Zero

$\Rightarrow$ A.S. also Diverges by LCT

O.S. Conditionally Convergent by Definition

$\boxed{\text{C.C.}}$

Note: If use CT on A.S then need

$$\frac{1}{9n-7} \ge \frac{1}{9n} \text{ and } \frac{1}{9}\sum_{n=1}^{\infty}\frac{1}{n}$$

Constant Multiple of Divergent Series $p=1$ is Divergent

4. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 3^n\, n!\, n^n}{n^3\,(2n)!}$

Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}\, 3^{n+1}\,(n+1)!\,(n+1)^{n+1}}{(n+1)^3\,[2(n+1)]!}}{\dfrac{\cancel{(-1)^n}\, 3^n\, n!\, n^n}{n^3\,(2n)!}} \right| \qquad \frac{n^3\,(2n)!}{3^n\, n!\, n^n}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{\cancel{3^n}} \cdot \frac{(n+1)!}{\cancel{n!}} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n^3}{(n+1)^3} \cdot \frac{\cancel{(2n)!}}{(2n+2)!}$$

$\cancel{3^n}\, 3$ $\qquad (n+1)\cancel{n!}$ $\qquad (n+1)^n (n+1)$ $\qquad \left(\frac{n \ \ 1/n}{n+1 \ \ 1/n}\right)^3$ $\qquad (2n+2)(2n+1)\cancel{(2n)!}$

$$= \lim_{n\to\infty} 3 \cdot \underbrace{\frac{(n+1)^n}{n^n}}_{e} \cdot \left(\frac{1}{1+\cancel{1/n}^{\,0}}\right)^{3 \to 1} \cdot \frac{\cancel{(n+1)}(n+1)\ 1/n}{(2n+2)(2n+1)\ 1/n}$$

$2\cancel{(n+1)}$

$$= \lim_{n\to\infty} \frac{3e}{2} \left(\frac{1+\cancel{1/n}^{\,0}}{2+\cancel{1/n}^{\,0}}\right)_{\to 1/2} = \frac{3e}{4} > 1$$

O.S. Diverges by R.T.

$e \approx 2.7$

$3e > 6$

4. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Name any convergence test(s) you use, and justify all of your work.

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^7+9}{n^9+7}$ $\xrightarrow[\text{must}]{\text{A.S.}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{n^7+9}{n^9+7} \approx \sum_{n=1}^{\infty} \frac{n^7}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

Converges, p-Series
$p = 2 > 1$

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^7+9}{n^9+7}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^9+9n^2}{n^9+7}\frac{\,^1/_{n^9}}{\,^1/_{n^9}} = \lim_{n\to\infty} \frac{1+\cancel{9/n^7}^{\to 0}}{1+\cancel{7/n^9}^{\to 0}} = 1$$

Finite, Non-Zero

$\Rightarrow$ A.S. also Converges by LCT

$\Rightarrow$ O.S. Absolutely Convergent by Definition

$\boxed{\text{A.C.}}$