Exam 1 Fall 2022 Answer Key

1(a)
$$\lim_{x\to 0} \frac{\cos(3x) - \arctan(2x) + 2x - 1}{e^{-4x} - 1 + 4x} \overset{0/0}{\underset{\text{L'H}}{=}} \lim_{x\to 0} \frac{-3\sin(3x) - \dfrac{2}{1+(2x)^2} + 2}{-4e^{-4x} + 4} \quad \frac{0}{0}$$

$$\underset{\text{prep}}{=} \lim_{x\to 0} \frac{-3\sin(3x) - 2(1+4x^2)^{-1} + 2}{-4e^{-4x} + 4}$$

$$\underset{\text{L'H}}{=} \lim_{x\to 0} \frac{-9\cos(3x) + 2(1+4x^2)^{-2}\cdot(8x)}{16e^{-4x}}$$

$$\underset{\text{rewrite}}{=} \lim_{x\to 0} \frac{-9\underbrace{\cos(3x)}_{\to 1} + \underbrace{\dfrac{16x}{(1+4x^2)^2}}_{\to 0}}{16\underbrace{e^{-4x}}_{\to 1}} = \boxed{-\frac{9}{16}}$$

1(b)
$$\lim_{x\to 0^+} x\ln x \;\; \overset{0\cdot(-\infty)}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{-\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = \boxed{0}$$

1(c)
$$\lim_{x\to\infty}\left(1 - \arcsin\left(\frac{2}{x^6}\right)\right)^{x^6} = e^{\lim_{x\to\infty} \ln\left(\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)^{x^6}\right)}$$

$$= e^{\lim_{x\to\infty} x^6 \cdot \ln\left(1 - \arcsin\left(\frac{2}{x^6}\right)\right)} \quad \infty\cdot 0$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(1-\arcsin\left(\frac{2}{x^6}\right)\right)}{\frac{1}{x^6}}} \quad \frac{0}{0}$$

$2x^{-6}$, $\frac{1}{x^6} \to -6x^{-7}$

$$= e^{\lim_{x\to\infty} \frac{\frac{1}{1-\arcsin\left(\frac{2}{x^6}\right)}\cdot\frac{-1}{\sqrt{1-\left(\frac{2}{x^6}\right)^2}}\cdot\left(\frac{-12}{x^7}\right)}{\frac{-6}{x^7}}}$$

$$= e^{1\cdot(-1)\cdot 2} = \boxed{e^{-2}}$$

2. $$\int_{-2}^{2}\sqrt{4-x^2}\,dx = \int_{x=-2}^{x=2}\cancel{\sqrt{4-4\sin^2\theta}}^{\,2\cos\theta}\cdot 2\cos\theta\,d\theta = 4\int_{x=-2}^{x=2}\cos^2\theta\,d\theta$$

$$\sqrt{4(1-\sin^2\theta)}$$

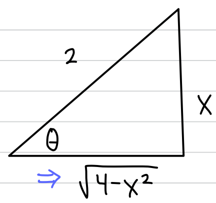
$$\sqrt{4\cos^2\theta}$$

Trig. Sub $\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}}$

$\sin\theta = \dfrac{x}{2}$

$\theta = \arcsin\left(\dfrac{x}{2}\right)$

$$= 4\int_{x=-2}^{x=2}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{4}{2}\int_{x=-2}^{x=2}1+\cos(2\theta)\,d\theta$$



$$= 2\left(\theta + \frac{\overbrace{\sin(2\theta)}^{\cancel{2}\sin\theta\cos\theta}}{\cancel{2}}\right)\Bigg|_{x=-2}^{x=2} = 2\left(\arcsin\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)\frac{\sqrt{4-x^2}}{2}\right)\Bigg|_{-2}^{2}$$

$$= 2\left(\cancel{\arcsin\left(\cancel{\tfrac{2}{2}}^{\,1}\right)}^{\frac{\pi}{2}} + \left(\frac{2}{2}\right)\cancel{\frac{\sqrt{4-4}}{2}}^{\,0} - \left(\cancel{\arcsin\left(\cancel{\tfrac{-2}{2}}^{\,-1}\right)}^{-\frac{\pi}{2}} + \left(\frac{-2}{2}\right)\cancel{\frac{\sqrt{4-4}}{2}}^{\,0}\right)\right)$$

$$= 2\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 2\pi$$

3. $$\int_{0}^{\ln\sqrt{3}}\frac{e^x}{\sqrt{4-e^{2x}}}\,dx = \int_{0}^{\ln\sqrt{3}}\frac{e^x}{\sqrt{4-(e^x)^2}}\,dx = \int_{1}^{\sqrt{3}}\frac{1}{\sqrt{4-u^2}}\,du = \arcsin\left(\frac{u}{2}\right)\Bigg|_{1}^{\sqrt{3}}$$

$\boxed{\begin{aligned} u &= e^x \\ du &= e^x\,dx \end{aligned}}$ $\boxed{\begin{aligned} x=0 &\Rightarrow u=e^0=1 \\ x=\ln\sqrt{3} &\Rightarrow u=e^{\ln\sqrt{3}}=\sqrt{3} \end{aligned}}$

$$= \cancel{\arcsin\left(\frac{\sqrt{3}}{2}\right)}^{\frac{\pi}{3}} - \cancel{\arcsin\left(\frac{1}{2}\right)}^{\frac{\pi}{6}}$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

4. $$\int_{e}^{e^3}\frac{1}{x\left(3+(\ln x)^2\right)}\,dx = \int_{1}^{3}\frac{1}{3+u^2}\,du = \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{1}^{3}$$

$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$ $\boxed{\begin{aligned} x=e &\Rightarrow u=\ln e=1 \\ x=e^3 &\Rightarrow u=\ln e^3=3 \end{aligned}}$

$$= \frac{1}{\sqrt{3}}\left(\cancel{\arctan\left(\cancel{\frac{3}{\sqrt{3}}}^{\sqrt{3}}\right)}^{\frac{\pi}{3}} - \cancel{\arctan\left(\frac{1}{\sqrt{3}}\right)}^{\frac{\pi}{6}}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{6\sqrt{3}}$$

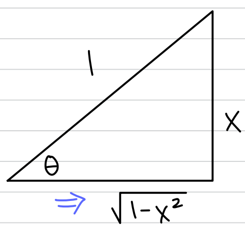
5. $$\int x\arcsin x\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot \cancel{\cos\theta}\,d\theta$$

$$\sqrt{1-\sin^2\theta} \to \sqrt{\cos^2\theta} \to \cancel{\cos\theta}$$

$$\boxed{\begin{array}{ll} u=\arcsin x & dv = x\,dx \\ du=\frac{1}{\sqrt{1-x^2}}\,dx & v=\frac{x^2}{2}\end{array}}$$

$$\boxed{\begin{array}{l} x=\sin\theta \\ dx=\cos\theta\,d\theta\end{array}}$$

$\theta = \arcsin x$



$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta\,d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2}\,dx$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\theta - \frac{\sin(2\theta)}{\cancel{2}}\right) + C$$

$\cancel{2}\sin\theta\cos\theta$

$$\boxed{= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left(\arcsin x - x\sqrt{1-x^2}\right) + C}$$

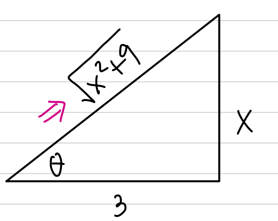
OR// $$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C$$

6. $$\int \frac{1}{(9+x^2)^{7/2}}\,dx = \int \frac{1}{\left(\sqrt{9+x^2}\right)^7}\,dx = \int \frac{1}{\left(\sqrt{9+9\tan^2\theta}\right)^7}\cdot 3\sec^2\theta\,d\theta$$

$$\left(\sqrt{9+9\tan^2\theta}\right)^7 \to \left(\sqrt{9(1+\tan^2\theta)}\right)^7 \to \left(\sqrt{9\sec^2\theta}\right)^7 \to (3\sec\theta)^7$$

$$\boxed{\begin{array}{l} x=3\tan\theta \\ dx = 3\sec^2\theta\,d\theta\end{array}}$$

$\tan\theta = \frac{x}{3}$



$$= \int \frac{1}{3^7\cdot\sec^7\theta}\cdot 3\sec^2\theta\,d\theta = \frac{1}{3^6}\int\frac{1}{\sec^5\theta}\,d\theta$$

$$= \frac{1}{729}\int \cos^5\theta\,d\theta = \frac{1}{729}\int \cos^4\theta\cdot\cos\theta\,d\theta = \frac{1}{729}\int \left(1-\sin^2\theta\right)^2\cdot\cos\theta\,d\theta$$

$(\cos^2\theta)^2$

$$= \frac{1}{729}\int (1-u^2)^2\,du = \frac{1}{729}\int 1-2u^2+u^4\,du = \frac{1}{729}\left(u-\frac{2}{3}u^3+\frac{u^5}{5}\right)+C$$

$$\boxed{\begin{array}{l} u=\sin\theta \\ du=\cos\theta\,d\theta\end{array}}$$

$$= \frac{1}{729}\left(\sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta\right)+C$$

$$= \frac{1}{729}\left(\frac{x}{\sqrt{x^2+9}} - \frac{2}{3}\left(\frac{x}{\sqrt{x^2+9}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{x^2+9}}\right)^5\right)+C$$

7. $\int x^7 \cdot \ln(x^3)\,dx = \frac{x^8}{8} \cdot \ln(x^3) - \frac{3}{8}\int \frac{\cancel{x^8}^{\,x^7}}{\cancel{x}}\,dx$

$u = \ln(x^3)$	$dv = x^7\,dx$
$du = \frac{1}{\cancel{x^3}_{\,x}} \cdot (3\cancel{x^2})\,dx$	$v = \frac{x^8}{8}$

$= \frac{x^8}{8}\ln(x^3) - \frac{3}{8} \cdot \frac{x^8}{8} + C$

$= \boxed{\frac{x^8}{8}\ln(x^3) - \frac{3}{64}x^8 + C}$