Ratio Test

Given a series $\sum_{n=1}^{\infty} a_n$. Consider $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, if

1. $L < 1$ then the Original Series is **Absolutely** Convergent
2. $L > 1$ then the Original Series Diverges
3. $L = 1$ INCONCLUSIVE

USED: On terms with both positive and negative terms. You can run the Ratio Test on the O.S. You do **not** need to waste your time shifting to the Absolute Series here. If the series is a candidate for the RT go straight to testing the limit on the O.S.

GOOD FOR: Series with terms containing Exponentials **and/or** Factorials. The terms may also contain logs or polynomials pieces WITH either exponentials or logs.

NOT GOOD FOR: Series with terms containing **just** logs **and/or** polynomials. Probably the ACT with IT/CT/LCT would work better for those cases.

WARNING: Do not get a declaration of Conditionally Convergent out of RT here. Not possible. The series either Absolutely Converges or simply Diverges, or is Inconclusive. No other options.

NOTE: If the Original Series is already equal to the Absolute series, then Absolute Convergence is the same as Convergence.

APPROACH:

- Given the original series, set up the Limit of the Absolute Value of the Ratio of successive terms. The order is important here, unlike with the LCT limit. Stack the next term on top of the previous term, that is $\frac{a_{n+1}}{a_n}$.
- Manage the justification of the limit carefully. Use algebra to flip the fractions. Try to repartner like terms and manage the limit of the product pieces individually (thanks to the Limit Laws).
- Compare L to the value 1.
  - If $L < 1$ then the original series is A.C. DONE. That is a strong result.
  - If $L > 1$ then the original series Diverges. DONE.
  - If $L = 1$ then this is NOT your convergence test.

EXAMPLES: Determine and state whether each of the following series **Converges** or **Diverges**. Name any convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ n!}{n^2\ 7^n}$ Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}\ (n+1)!}{(n+1)^2\ 7^{n+1}}}{\dfrac{(-1)^n\ n!}{n^2\ 7^n}}\right| = \lim_{n\to\infty}\left(\frac{n^2}{(n+1)^2}\right)\left(\frac{7^n}{7^{n+1}}\right)\left(\frac{(n+1)!}{n!}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\,\frac{\frac{1}{n}}{\frac{1}{n}}\right)^2\left(\frac{\cancel{7^n}}{\cancel{7^n}\cdot 7}\right)\left(\frac{(n+1)\cancel{n!}}{\cancel{n!}}\right) = \lim_{n\to\infty}\cancel{\left(\frac{1}{1+\frac{1}{n}\to 0}\right)^2}\to 1\left(\frac{1}{7}\right)(n+1)$$

$$= \lim_{n\to\infty}(1)\frac{\cancel{n}\to\infty + 1}{7} = \infty > 1 \text{ Therefore, the O.S. } \boxed{\text{Diverges by the RT}}.$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (\ln n)\ (3n)!}{n^n\ 2^{4n}\ (n!)^2}$ Try Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}\ \ln(n+1)\ (3(n+1))!}{(n+1)^{n+1}\ 2^{4(n+1)}\ ((n+1)!)^2}}{\dfrac{(-1)^n\ \ln n\ (3n)!}{n^n\ 2^{4n}\ (n!)^2}}\right|$$

$$= \lim_{n\to\infty}\left(\frac{(3n+3)!}{(3n)!}\right)\left(\frac{n^n}{(n+1)^{n+1}}\right)\left(\frac{(n!)^2}{((n+1)!)^2}\right)\left(\frac{2^{4n}}{2^{4n+4}}\right)\left(\frac{\ln(n+1)}{\ln n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{\cancel{(3n)!}}\right)\left(\frac{n^n}{(n+1)^n(n+1)}\right)\left(\frac{(n!)^2}{(n+1)^2(n!)^2}\right)\left(\frac{\cancel{2^{4n}}}{\cancel{2^{4n}}2^4}\right)\cancel{\left(\frac{\ln(n+1)}{\ln n}\right)}\to 1^{(*)}$$

$$\stackrel{(*)}{=} \lim_{n\to\infty}\left(\frac{3\cancel{(n+1)}(3n+2)(3n+1)}{1}\right)\left(\frac{1}{e}\right)\left(\frac{1}{\cancel{n+1}}\right)\left(\frac{1}{(n+1)^2}\right)\left(\frac{1}{16}\right)(1)$$

$$= \lim_{n\to\infty}\left(\frac{3}{16e}\right)\left(\frac{3n+2}{n+1}\,\frac{(\frac{1}{n})}{(\frac{1}{n})}\right)\left(\frac{3n+1}{n+1}\,\frac{(\frac{1}{n})}{(\frac{1}{n})}\right) = \lim_{n\to\infty}\left(\frac{3}{16e}\right)\left(\frac{3+\dfrac{2}{n}}{1+\dfrac{1}{n}}\right)\left(\frac{3+\dfrac{1}{n}}{1+\dfrac{1}{n}}\right) = \frac{27}{16e} < 1$$

Therefore the original series $\boxed{\text{Converges Absolutely by the Ratio test}}$.

$$(*) = \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}^{\frac{\infty}{\infty}} = \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \stackrel{\text{L'H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+1}}{\dfrac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} \stackrel{\frac{\infty}{\infty}\ \text{L'H}}{=} \lim_{x\to\infty}\frac{1}{1} = 1$$