Math 121 Final Exam Spring 2021

1. $$\lim_{x\to\infty}\left[1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right]^{x^4} \qquad 1^\infty$$

$$= e^{\lim\limits_{x\to\infty} \ln\left[\left(1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right)^{x^4}\right]}$$

$\infty \cdot 0$

$$= e^{\lim\limits_{x\to\infty} x^4 \ln\left(1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right)}$$

$$= e^{\lim\limits_{x\to\infty} \frac{\ln\left(1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)\right)}{\frac{1}{x^4}}} \qquad 0/0$$

x^{-4}

L'H

$$= e^{\lim\limits_{x\to\infty} \frac{\frac{1}{1-\arcsin\left(\frac{3}{x^4}\right)-\sin\left(\frac{1}{x^4}\right)}\left[-\frac{1}{\sqrt{1-\left(\frac{3}{x^4}\right)^2}}\left(\frac{-12}{x^5}\right)-\cos\left(\frac{1}{x^4}\right)\left(\frac{-4}{x^5}\right)\right]}{-4/x^5}}$$

$3x^{-4}$ $\quad 1 \quad -1 \quad 1 \quad 3 \quad -1 \quad 0$ $\quad \cancel{-\frac{x^5}{4}}$

$$= e^{1(-3-1)} = e^{-4}$$

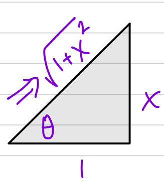
2(a) $$\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{7/2}}\,dx = \int_0^{\sqrt{3}} \frac{1}{\left(\sqrt{1+x^2}\right)^7}\,dx = \int_{x=0}^{x=\sqrt{3}} \frac{1}{\left(\sqrt{1+\tan^2\theta}\right)^7}\cdot \sec^2\theta\, d\theta$$

$\theta = \pi/3$ $\quad$ or $\theta = 0$

$\sqrt{\sec^2\theta}$ $\quad (\sec\theta)^7$ $\quad \sec^5\theta$

Trig. Sub

$\boxed{x = \tan\theta \quad dx = \sec^2\theta\, d\theta}$



$$= \int_{x=0}^{x=\sqrt{3}} \frac{1}{\sec^5\theta}\,d\theta = \int_{x=0}^{x=\sqrt{3}} \cos^5\theta\, d\theta = \int_{x=0}^{x=\sqrt{3}} \cos^4\theta \cdot \cos\theta\, d\theta$$

ODD Power $\quad (\cos^2\theta)^2$

$\boxed{u = \sin\theta \quad du = \cos\theta\, d\theta}$

$$= \int_{x=0}^{x=\sqrt{3}} (1-\sin^2\theta)^2 \cdot \cos\theta\, d\theta = \int_{x=0}^{x=\sqrt{3}} (1-u^2)^2\,du = \int_{x=0}^{x=\sqrt{3}} 1-2u^2+u^4\,du$$

$u \quad du$

$$= u - \frac{2}{3}u^3 + \frac{u^5}{5}\Big|_{x=0}^{x=\sqrt{3}} = \sin\theta - \frac{2}{3}\sin^3\theta + \frac{\sin^5\theta}{5}\Big|_{x=0}^{x=\sqrt{3}}$$

$$= \frac{x}{\sqrt{1+x^2}} - \frac{2}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3 + \frac{1}{5}\left(\frac{x}{\sqrt{1+x^2}}\right)^5 \Bigg|_0^{\sqrt{3}}$$

$$= \left(\frac{\sqrt{3}}{\cancel{\sqrt{4}}\,2} - \frac{2}{3}\left(\frac{\sqrt{3}}{\cancel{\sqrt{4}}\,2}\right)^3 + \frac{1}{5}\left(\frac{\sqrt{3}}{\cancel{\sqrt{4}}\,2}\right)^5\right) - \cancel{\left(0 - 0 + 0\right)}\ 0$$

$$= \frac{\sqrt{3}}{2} - \cancel{\frac{2}{3}}\cdot\frac{3\sqrt{3}}{\cancel{8}\,4} + \frac{1}{5}\cdot\frac{9\sqrt{3}}{32} = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} + \frac{9\sqrt{3}}{160}$$

$$= \frac{80\sqrt{3}}{160} - \frac{40\sqrt{3}}{160} + \frac{9\sqrt{3}}{160} = \frac{49\sqrt{3}}{160}$$ Match!

2(b) $$\int x^4 \arcsin x \, dx = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{x^5}{\sqrt{1-x^2}}\,dx$$

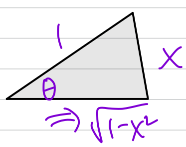
IBP

$\boxed{u = \arcsin x \quad dv = x^4 dx \quad du = \frac{1}{\sqrt{1-x^2}} \quad v = \frac{x^5}{5}}$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\underbrace{\sqrt{1-\sin^2\theta}}_{\sqrt{\cos^2\theta} = \cancel{\cos\theta}}}\cancel{\cos\theta}\,d\theta$$

Trig. Sub

$\boxed{x = \sin\theta \quad dx = \cos\theta\, d\theta}$



$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \underbrace{\sin^5\theta}_{\sin^4\theta\ \sin\theta,\ (\sin^2\theta)^2}\,d\theta$$ ODD Power Again

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int (1-\cos^2\theta)^2 \sin\theta\,d\theta$$

$\boxed{u = \cos\theta \quad du = -\sin\theta\,d\theta \quad -du = \sin\theta\,d\theta}$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int (1-u^2)^2\,du$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int 1 - 2u^2 + u^4\,du$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\overset{\cos\theta}{u} - \frac{2}{3}u^3 + \frac{u^5}{5}\right] + C$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\overset{\sqrt{1-x^2}}{\cos\theta} - \frac{2}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right] + C$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left[\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 + \frac{1}{5}\left(\sqrt{1-x^2}\right)^5\right] + C$$

2(c)

$$\int_0^{\sqrt{3}} (x+3)\arctan x\,dx = \left(\frac{x^2}{2}+3x\right)\arctan x\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{\frac{x^2}{2}+3x}{1+x^2}\,dx$$

split

IBP

$u = \arctan x$	$dv = x+3\,dx$
$du = \frac{1}{1+x^2}\,dx$	$v = \frac{x^2}{2}+3x$

$$= \left(\frac{x^2}{2}+3x\right)\arctan x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2+1-1}{1+x^2}\,dx - 3\int_0^{\sqrt{3}} \frac{x}{1+x^2}\,dx$$

slip-in | slip-out

split - split

$$-\frac{1}{2}\int_0^{\sqrt{3}} \cancel{\frac{x^2+1}{1+x^2}}^{\,1} - \frac{1}{1+x^2}\,dx$$

$$-\frac{1}{2}\left[x - \arctan x\right]\Big|_0^{\sqrt{3}}$$

$$= \left(\frac{x^2}{2}+3x\right)\arctan x\Big|_0^{\sqrt{3}} - \frac{x}{2} + \frac{\arctan x}{2}\Big|_0^{\sqrt{3}} - \frac{3}{2}\ln|1+x^2|\Big|_0^{\sqrt{3}}$$

or pull limits of integration off all the way to the end!

$$= \left(\frac{3}{2}+3\sqrt{3}\right)\cancelto{\pi/3}{\arctan\sqrt{3}} - 0 - \frac{\sqrt{3}}{2} + \frac{\cancelto{\pi/3}{\arctan\sqrt{3}}}{2} - \left(0 + \cancelto{0}{\arctan 0}\right) - \frac{3}{2}\left(\ln 4 - \cancelto{0}{\ln 1}\right)$$

$$= \frac{\cancel{3}}{2}\left(\frac{\pi}{\cancel{3}}\right) + \cancel{3}\sqrt{3}\left(\frac{\pi}{\cancel{3}}\right) - \frac{\sqrt{3}}{2} + \frac{\pi}{6} - \frac{3}{2}\ln 4$$

$$4^{3/2} = \left(\sqrt[2]{4}\right)^3 = 8$$

$$= \frac{\pi}{2} + \sqrt{3}\pi - \frac{\sqrt{3}}{2} + \frac{\pi}{6} - \ln\left[\cancelto{8}{4^{3/2}}\right]$$

$2\pi/3$

$$\pi/2 + \pi/6$$

$$= 3\pi/6 + \pi/6 = 4\pi/6 = 2\pi/3$$

$$= \left(\frac{2}{3}+\sqrt{3}\right)\pi - \frac{\sqrt{3}}{2} - \ln 8$$

Match!

3(a) $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2-6x+25}\,dx = \lim_{t\to-\infty}\int_t^{-1}\frac{1}{x^2-6x+25}\,dx$

Improper

Quadratic Irreducible

$b^2-4ac = 36-100 < 0$

$\ominus$

$(x-3)^2+16$

$(x-3)^2 = x^2-6x+9$

Complete Square

$\displaystyle = \lim_{t\to-\infty}\int_t^{-1}\frac{1}{(x-3)^2+16}\,dx$

$\boxed{\begin{array}{l} u = x-3 \\ du = dx \end{array}}$

$\boxed{\begin{array}{l} x=t \Rightarrow u = t-3 \\ x=-1 \Rightarrow u=-4 \end{array}}$

$\displaystyle = \lim_{t\to-\infty}\int_{t-3}^{-4}\frac{1}{u^2+16}\,du$

a-rule

$\displaystyle = \lim_{t\to-\infty}\frac{1}{4}\arctan\left(\frac{u}{4}\right)\Bigg|_{t-3}^{-4}$

$\displaystyle = \lim_{t\to-\infty}\frac{1}{4}\left(\arctan\left(\frac{-4}{4}\right) - \arctan\left(\frac{t-3}{4}\right)\right)$

$-\frac{\pi}{4}$ $\quad -1 \quad \ominus\frac{\pi}{2} \quad -\infty$

$\displaystyle = \frac{1}{4}\left(-\frac{\pi}{4}+\frac{\pi}{2}\right) = \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{\pi}{16}$ Converges

3(b) $\displaystyle\int_{-1}^{6}\frac{15-x}{x^2-6x-7}\,dx = \int_{-1}^{6}\frac{15-x}{(x-7)(x+1)}\,dx = \lim_{t\to-1^+}\int_t^6\frac{15-x}{(x-7)(x+1)}\,dx$

Factors

Improper

PFD:

$\displaystyle \cancel{(x-7)(x+1)}\left[\frac{15-x}{\cancel{(x-7)(x+1)}} = \frac{A}{x-7}+\frac{B}{x+1}\right](x-7)(x+1)$

$15-x = A(x+1)+B(x-7)$

$= Ax + A + Bx - 7B$

$= (A+B)x + (A-7B)$

Conditions:

- $A+B=-1 \Rightarrow A=-B-1$
- $A-7B=15$

$-B-1-7B=15$

$-8B=16$

$B=-2 \Rightarrow A=1$

PFD

$\displaystyle = \lim_{t\to-1^+}\int_t^6 \frac{1}{x-7}-\frac{2}{x+1}\,dx$

$\displaystyle = \lim_{t\to-1^+}\ln|x-7| - 2\ln|x+1|\Big|_t^6$

$\displaystyle = \lim_{t\to-1^+}\ln 1 - 2\ln 7 - \left(\ln|t-7| - 2\ln|t+1|\right)$

$0 \quad |-8| \quad 8 \quad -1 \quad -1^+ \quad 0^+ \quad -\infty$

Finite $\quad \ln 8$ Finite

$= -\infty$ Diverges

3(c) $\displaystyle\int_{0}^{e} \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^e \ln x \cdot x^{-1/2}\,dx$

Improper

IBP

$u = \ln x$	$dv = x^{-1/2}\,dx$
$du = \frac{1}{x}\,dx$	$v = 2\sqrt{x}$

$$= \lim_{t\to 0^+} 2\sqrt{x}\ln x \Big|_t^e - 2\int_t^e \frac{\sqrt{x}}{x}\,dx$$

$x^{-1/2}$

$$= \lim_{t\to 0^+} 2\sqrt{x}\ln x \Big|_t^e - 4\sqrt{x}\Big|_t^e$$

$$= \lim_{t\to 0^+} 2\sqrt{e}\,\underbrace{\ln e}_{1} - \underbrace{2\sqrt{t}\ln t}_{0\cdot(-\infty)\to 0} - 4\sqrt{e} + 4\underbrace{\sqrt{t}}_{\to 0}$$

See ○

$$= 2\sqrt{e} - 4\sqrt{e} = -2\sqrt{e} \qquad \text{Converges}$$

○ $\displaystyle\lim_{t\to 0^+} \underbrace{\sqrt{t}\ln t}_{0\cdot(-\infty)} = \lim_{t\to 0^+} \frac{\ln t}{1/\sqrt{t}} \overset{-\infty/\infty}{\underset{\text{L'H}}{=}} \lim_{t\to 0^+} \left(\frac{\frac{1}{t}}{\frac{-1}{2t^{3/2}}}\right) = \lim_{t\to 0^+} \underbrace{-2\sqrt{t}}_{\to 0} = 0$

$-2t^{3/2}$

3(d) $\displaystyle\int_0^{1/2} \frac{1}{x\ln x}\,dx = \lim_{t\to 0^+} \int_t^{1/2} \frac{1}{x\ln x}\,dx$

Improper

⋮ u-sub if needed

$$= \lim_{t\to 0^+} \ln|\ln x| \Big|_t^{1/2}$$

$$= \lim_{t\to 0^+} \underbrace{\ln|\ln \tfrac{1}{2}|}_{\text{Finite}} - \ln|\ln t|$$

$t \to 0^+$, $|-\infty|$, ∞, ∞

$$= -\infty \qquad \text{Diverges}$$

4(a) $\sum_{n=1}^{\infty} \cos^2\left(\frac{\pi n^6 + 2021}{6n^6+1}\right)$ Diverges by nTDT because

$$\lim_{n\to\infty} \cos^2\left(\frac{\cancel{\pi n^6} + 2021}{\cancel{6n^6}+1}\right) = \left[\cancel{\cos}\left(\frac{\pi}{6}\right)\right]^{2} = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4} \neq 0$$

$\frac{\pi}{6}$ $\frac{\sqrt{3}}{2}$

pass limit

4(b) $\sum_{n=1}^{\infty} \frac{(-1)^n \cos^2(\pi n^6 + 2021)}{6n^6+1} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{\cos^2(\pi n^6 + 2021)}{6n^6+1} \approx \sum \frac{1}{n^6}$

Converges p-Series $p=6>1$

Bound Terms

$$\frac{\cos^2(\pi n^6 + 2021)}{6n^6+1} \leq \frac{1}{6n^6+1} \leq \frac{1}{6n^6} \leq \frac{1}{n^6}$$

$\Rightarrow$ A.S. also Converges by C.T.

$\Rightarrow$ O.S. A.C. by Definition

4(c) $\sum_{n=1}^{\infty} \underbrace{\frac{\ln(2021)}{n^6}}_{\oplus \text{ terms}} = \underbrace{\ln(2021)}_{\text{constant}} \sum_{n=1}^{\infty} \frac{1}{n^6}$

Note: O.S = A.S.

Constant Multiple of Convergent p-Series $p=6>1$ is Convergent

$\Rightarrow$ O.S. is also A.C. because the original given series is already the same as the Absolute Series

4(d) $\sum_{n=1}^{\infty} \frac{n^6}{\ln(n+2021)}$ Diverges by nTDT because

$$\lim_{n\to\infty} \frac{n^6}{\ln(n+2021)} = \lim_{x\to\infty} \frac{x^6}{\ln(x+2021)} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{6x^5}{1/\cancel{2021}} = \lim_{x\to\infty} 6x^5(x+2021) = \infty \neq 0$$

5(a) $\frac{5}{3} - 1 + \frac{5}{7} - \frac{5}{9} + \frac{5}{11} - \ldots = \frac{5}{3} - \frac{5}{5} + \frac{5}{7} - \frac{5}{9} + \frac{5}{11} - \ldots$

$$= 5\left[\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \ldots\right]$$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

$$= -5\left[-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right]$$

$$= -5\left[(\cancel{\arctan 1}) - 1\right]$$

$\pi/4$

$$= -5\left[\pi/4 - 1\right] = 5 - \frac{5\pi}{4}$$

5(b) $\frac{1}{2} - \frac{1}{8} + \frac{1}{3\cdot 2^3} - \frac{1}{64} + \frac{1}{5\cdot 2^5} - \cdots = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} - \cdots$

2^3 $\quad$ 4^3

$$= \left(\frac{1}{2}\right) - \frac{(1/2)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} - \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \ln(1 + 1/2) = \ln(3/2)$$

5(c) $\sum_{n=0}^{\infty} \frac{(-3)^n - 2}{4^n} = \sum_{n=0}^{\infty} \frac{(-3)^n}{4^n} - \frac{2}{4^n} = \sum_{n=0}^{\infty} \frac{(-3)^n}{4^n} - 2\sum_{n=0}^{\infty} \frac{1}{4^n}$ $\quad$ Both Geometric.

Split-Split

$$= \sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n - 2\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

or more generally

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

$$= \frac{1}{1-(-3/4)} - 2\left(\frac{1}{1-1/4}\right)$$

$+$ $\quad$ $7/4$ $\quad$ $3/4$ $\quad$ a

$$= 4/7 - 2(4/3) = 4/7 - 8/3 = \frac{12}{21} - \frac{56}{21} = -\frac{44}{21}$$

5(d) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{\boxed{2n+1}}}{(\sqrt{2})^{4n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{((\sqrt{2})^{\cancel{2}})^{2n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n}(2n)!}$

2

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \pi \sum_{n=0}^{\infty} \frac{(-1)^n (\pi/2)^{2n}}{(2n)!} = \pi \cancel{\cos(\pi/2)}^{\,0} = 0$$

5(e) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2^4)\pi^{2n}}{2^{4n}(2n)!} = -16 \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2^2)^{2n}(2n)!} = -16 \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^{2n}(2n)!}$

$$= -16 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = -16\, \cancel{\cos(\pi/4)}^{\sqrt{2}/2} = -8\sqrt{2}$$

5(f) $-\pi + \frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \frac{\pi^7}{7!} - \frac{\pi^9}{9!} + \dots = -\left[\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \dots\right]$

$= -\cancel{\sin\pi}^{\,0} = 0$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

5(g) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\ln 9)^n}{2^{n+1}\, n!} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (\ln 9)^n}{2^n \cdot n!} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 9}{2}\right)^n}{n!}$

$= -\frac{1}{2} e^{\left(\frac{-\ln 9}{2}\right)} = -\frac{1}{2} e^{\left(-\frac{1}{2}\right)\cdot \ln 9}$

$= -\frac{1}{2} \cancel{e^{\ln}}(9^{-1/2}) = -\frac{1}{2} \cdot \frac{1}{\cancel{\sqrt{9}}_{\,3}} = -\frac{1}{2} \cdot \frac{1}{3} = -\frac{1}{6}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)(5x-2)^n}{n^2\, 8^n}$

Ratio Test

$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}}(n+2)(5x-2)^{n+1}}{(n+1)^2\, 8^{n+1}}}{\frac{\cancel{(-1)^n}(n+1)(5x-2)^n}{n^2 \cdot 8^n}} \right| = \lim_{n\to\infty} \cancel{\frac{n+2}{n+1}}^{\,1} \left|\cancel{\frac{(5x-2)^{n+1}}{(5x-2)^n}}\right| \cdot \frac{n^2}{(n+1)^2} \cdot \frac{\cancel{8^n}}{8^{n+1}}$

$|5x-2|$

$\cancel{\left(\frac{n}{n+1}\right)^2} \to 1$

Converges by R.T. when $\frac{1}{8}$

$= \frac{|5x-2|}{8} < 1$

$|5x-2| < 8$

$-8 < 5x-2 < 8$
$+2 \quad +2 \quad +2$
$-6 < 5x < 10$
$-6/5 < x < 2$

Manually Test Convergence at Endpoints (where $L=1$)

Take $x = -6/5$. Series becomes

$\hookrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)\left(\cancel{5}\left(-\frac{6}{\cancel{5}}\right) - 2\right)^n}{n^2 \cdot 8^n} = \sum_{n=1}^{\infty} \frac{\cancel{(-1)^{2n}}(n+1)\cancel{8^n}}{n^2 \cdot \cancel{8^n}}$

$(-1)^n 8^n$
$(-8)^n$

Even
1

$= \sum_{n=1}^{\infty} \frac{n+1}{n^2} \approx \sum_{n=0} \frac{1}{n}$ Diverges, Harmonic, p-Series $p=1$

Bound Terms:

$\frac{n+1}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$ $\Rightarrow$ Series also Diverges by C.T.

(or use LCT Limit too)

Take $x=2$. Series becomes

$$\hookrightarrow \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)\overset{\cancel{8^n}}{(5|2|-2)^n}}{n^2 \cdot \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{n^2}$$

AST 1. $b_n = \dfrac{n+1}{n^2} > 0$

2. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{n+1}{n^2} = 0$ ✓

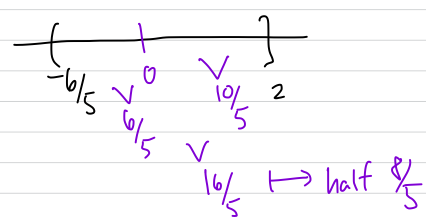
3. Terms Decreasing.

$$f(x) = \frac{x+1}{x^2} \text{ has } f'(x) = \frac{x^2(1)-(x+1)(2x)}{x^4} = \frac{\overset{\ominus}{-x^2-x}}{\underset{\oplus}{x^4}} < 0$$

$\Rightarrow$ Series Converges by A.S.T.

Finally, $I = \left[-\frac{6}{5}, 2\right]$

$R = 8/5$



6(b) $$\sum_{n=1}^{\infty} \frac{n^n (\ln n)(x-7)^n}{(2n)!\, e^n \sqrt{n}}$$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^{n+1} \ln(n+1)(x-7)^{n+1}}{(2(n+1))!\, e^{n+1}\sqrt{n+1}}}{\dfrac{n^n (\ln n)(x-7)^n}{(2n)!\, e^n \sqrt{n}}} \right|$$

$$= \lim_{n\to\infty} \frac{\overset{e\,\leftarrow\,(n+1)^n\,\cancel{(n+1)}}{(n+1)^{n+1}}}{n^n} \cdot \underbrace{\cancel{\frac{\ln(n+1)}{\ln n}}}_{\text{See }\bigcirc} \cdot \overset{1\quad |x-7|}{\left|\cancel{\frac{(x-7)^{n+1}}{(x-7)^n}}\right|} \cdot \frac{\cancel{(2n)!}}{\underset{\underset{2(n+1)}{(2n+2)(2n+1)\cancel{(2n)!}}}{(2n+2)!}} \cdot \underset{1/e}{\cancel{\frac{e^n}{e^{n+1}}}} \cdot \overset{\cancel{\sqrt{\frac{n}{n+1}}}\to 1}{\frac{\sqrt{n}}{\sqrt{n+1}}}$$

$$= \lim_{n\to\infty} \cancel{\frac{e}{e}} \cdot \underset{\to 0}{\cancel{\frac{|x-7|}{2(2n+1)}}} = 0 < 1$$

Converges by R.T. for all x.

$$\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}\overset{\infty/\infty}{=}\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x}\overset{\infty/\infty}{\underset{L'H}{=}}\lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{x}{x+1}\overset{\infty/\infty}{\underset{L'H}{=}}\lim_{x\to\infty}\frac{1}{x}=1 \checkmark$$

Finally, $I=(-\infty,\infty)$

$R=\infty$

7. $$\ln\left(\tfrac{3}{2}\right)=\ln\left(1+\tfrac{1}{2}\right)=\frac{1}{2}-\frac{\left(\frac{1}{2}\right)^2}{2}+\frac{\left(\frac{1}{2}\right)^3}{3}-\frac{\left(\frac{1}{2}\right)^4}{4}+\cdots$$

$\frac{1}{4}$ $\quad \frac{1}{8}$ $\quad \frac{1}{16}$

$$=\frac{1}{2}-\frac{1}{8}+\frac{1}{24}-\frac{1}{64}+\cdots$$

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

Estimate

$$\approx\frac{1}{2}-\frac{1}{8}+\frac{1}{24}=\frac{12}{24}-\frac{3}{24}+\frac{1}{24}=\frac{10}{24}=\frac{5}{12}$$

Using ASET we can Estimate the full Sum using only the first three terms with error at most $|\text{First Neglected Term}|=\frac{1}{64}<\frac{1}{50}$ as desired.

Fun Note: This is the Same Sum as above in 5(b).

8(a) $$\ln(x+3)=\int\frac{1}{x+3}\,dx=\int\frac{1}{3+x}\,dx=\int\frac{1}{3(1+x/3)}\,dx=\int\frac{1}{3(1-(-x/3))}\,dx$$

$$=\int\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n dx=\int\frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n}\,dx=\int\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^{n+1}}\,dx$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}+C \quad \ln 3$$

Expand

$n=0 \quad n=1 \quad n=2 \quad n=3$

$$\ln(x+3)=\frac{x}{3}-\frac{x^2}{3^2\cdot 2}+\frac{x^3}{3^3\cdot 3}-\cdots+C$$

Test $x=0$: $\ln 3=0-0+0-\cdots+C\Rightarrow C=\ln 3$ here

Finally, $\ln(x+3) = \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}$

8(b). $\ln(x+3) = \ln(3+x) = \ln\left(3\left(1+\frac{x}{3}\right)\right) = \ln 3 + \ln\left(1+\frac{x}{3}\right)$

Substitution here

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{3}\right)^{n+1}}{n+1} = \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}$$

Match!

Just for Fun: Chart Method / Definition of Maclaurin Series

$f(x) = \ln(x+3)$	$f(0) = \ln 3$
$f'(x) = \frac{1}{x+3} = (x+3)^{-1}$	$f'(0) = \frac{1}{3}$
$f''(x) = -(x+3)^{-2} = \frac{-1}{(x+3)^2}$	$f''(0) = \frac{-1}{3^2} = \frac{-1}{9}$
$f'''(x) = 2(x+3)^{-3} = \frac{2}{(x+3)^3}$	$f'''(0) = \frac{2}{3^3} = \frac{2}{27}$
$f^{(4)}(x) = -6(x+3)^{-4} = \frac{-6}{(x+3)^4}$	$f^{(4)}(0) = \frac{-6}{3^4} = \frac{-6}{81}$
$\vdots$	$\vdots$

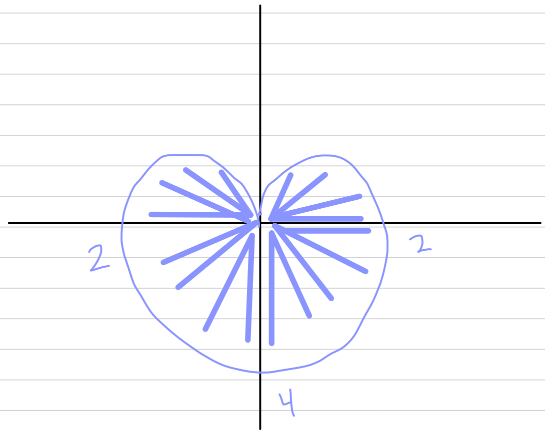
Maclaurin Series

$$\cancelto{\ln 3}{f(0)} + \cancelto{\frac{1}{3}}{f'(0)}x + \frac{\cancelto{-1/9}{f''(0)}}{2!}x^2 + \frac{\cancelto{\cancel{2}/27}{f'''(0)}}{3!_{\,3\cdot\cancel{2}}}x^3 + \frac{\cancelto{-\cancel{6}/81}{f^{(4)}(0)}}{4!_{\,4\cdot\cancel{3\cdot 2}}}x^4 + \cdots$$

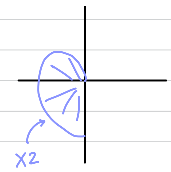
$$= \ln 3 + \frac{x}{3} - \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} - \frac{x^4}{3^4\cdot 4} + \cdots$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}\cdot(n+1)}$$ Match!

9. $r = 2 - 2\sin\theta$



OR use Double by Symmetry



$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\text{Polar Radius})^2 \, d\theta$$

OR Distribute $\frac{1}{2}$ works too.

$$= \frac{1}{2}\int_0^{2\pi} (2 - 2\sin\theta)^2 \, d\theta \quad = \frac{1}{2}\int_0^{2\pi} 4 - 8\sin\theta + 4\sin^2\theta \, d\theta$$

$$\overset{2}{\cancel{4}}\left[\frac{1-\cos(2\theta)}{\cancel{2}}\right] \quad \text{Half-Angle}$$

$$= \frac{1}{2}\int_0^{2\pi} \underbrace{4 - 8\sin\theta + 2}_{6} - 2\cos(2\theta) \, d\theta$$

$$= \frac{1}{2}\left[6\theta + 8\cos\theta - \sin(2\theta)\right]\Big|_0^{2\pi}$$

$$= \frac{1}{2}\left[6(2\pi) + 8\underbrace{\cos(2\pi)}_{1} - \underbrace{\sin(4\pi)}_{0} - \left(0 + 8\underbrace{\cos 0}_{1} - \underbrace{\sin 0}_{0}\right)\right]$$

$$= \frac{1}{2}\left[12\pi + \cancel{8} - \cancel{8}\right] = 6\pi$$

Extra Side Note: We know from class that the standard Cardioids $r = 1 \pm \cos\theta$ and $r = 1 - \sin\theta$ all have bounded are equaling $3\pi/2$.

Here $r = 2 - 2\cos\theta = 2(1 - \cos\theta)$ and the integral can be factored algebraically with a scaling effect of $2^2 = 4$

Here $\text{Area} = \frac{1}{2}\int_0^{2\pi} (2-2\sin\theta)^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} \left[2(1-\sin\theta)\right]^2 d\theta$

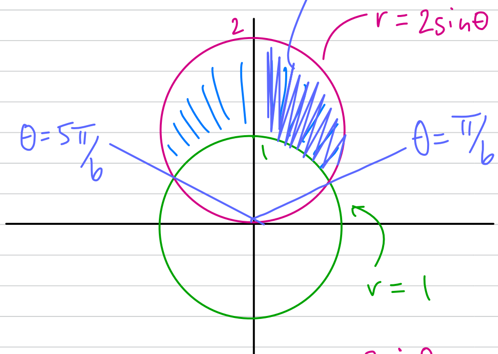
$$= \frac{1}{2}\int_0^{2\pi} 4 \cdot (1-\cos\theta)^2 \, d\theta = 4\left[\frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, d\theta\right] = 6\pi$$

$3\pi/2$

Match! Makes Sense.

10(a) $r = 1$ $\quad r = 2\sin\theta$

Double by Symmetry



Intersect?

$2\sin\theta = 1$

$\sin\theta = 1/2$

$\theta = \pi/6, \frac{5\pi}{6}$ (by symmetry)

$$\text{Area} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (\overset{2\sin\theta}{\text{Outer Polar Radius}})^2 - (\overset{1}{\text{Inner Polar Radius}})^2 \, d\theta$$

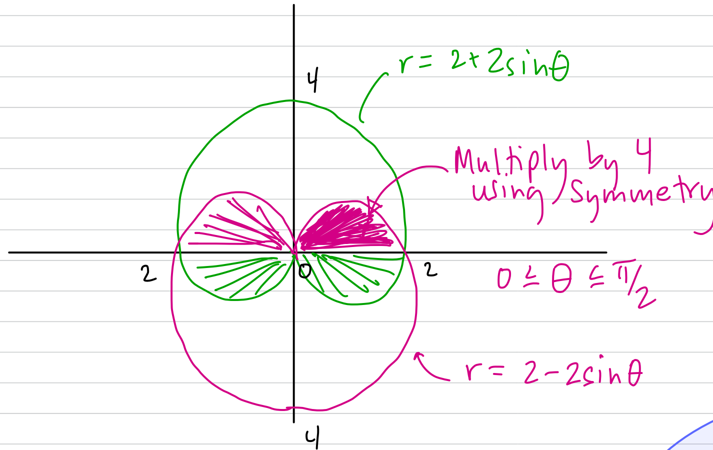
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2\sin\theta)^2 - 1^2 \, d\theta$$

Alternate Integral

$$= 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (2\sin\theta)^2 - 1^2 \, d\theta\right]$$

Double using Symmetry

10(b) $r = 2 + 2\sin\theta \qquad r = 2 - 2\sin\theta$



$$\text{Area} = 4\left[\frac{1}{2}\int_0^{\pi/2} (\underbrace{\text{Polar Radius}}_{2-2\sin\theta})^2\, d\theta\right] = \boxed{4\left[\frac{1}{2}\int_0^{\pi/2} (2-2\sin\theta)^2\, d\theta\right]}$$

using Symmetry

Alternate Integrals

$$2\left[\frac{1}{2}\int_0^{\pi} (2-2\sin\theta)^2\, d\theta\right]$$

OR

$$2\left[\frac{1}{2}\int_{\pi}^{2\pi} (2+2\sin\theta)^2\, d\theta\right]$$

OR

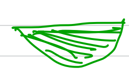
$$4\left[\frac{1}{2}\int_{\pi/2}^{\pi} (2-2\sin\theta)^2\, d\theta\right]$$

OR

$$4\left[\frac{1}{2}\int_{\pi}^{3\pi/2} (2+2\sin\theta)^2\, d\theta\right]$$

OR

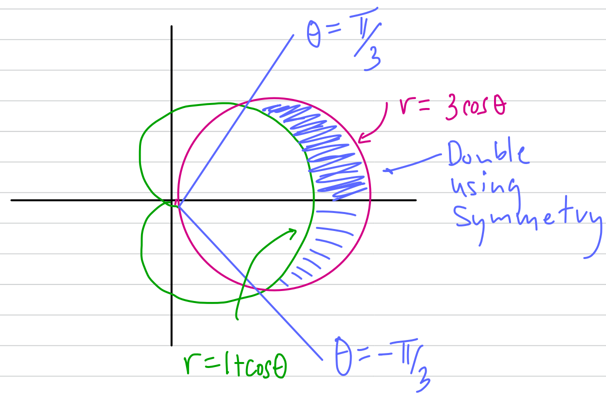
$$4\left[\frac{1}{2}\int_{3\pi/2}^{2\pi} (2+2\sin\theta)^2\, d\theta\right]$$



10(c) $r = 1+\cos\theta$ $\quad r = 3\cos\theta$

Intersect?

$$1+\cos\theta = 3\cos\theta$$
$$1 = 2\cos\theta$$
$$\cos\theta = 1/2$$
$$\theta = \pm \pi/3$$



$$\text{Area} = \frac{1}{2}\int_{-\pi/3}^{\pi/3} \left(\text{Outer } \overset{3\cos\theta}{\cancel{\text{Polar Radius}}}\right)^2 - \left(\text{Inner } \overset{1+\cos\theta}{\cancel{\text{Polar Radius}}}\right)^2 d\theta$$

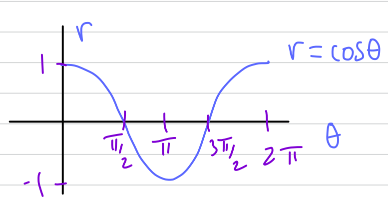
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (3\cos\theta)^2 - (1+\cos\theta)^2 \, d\theta$$

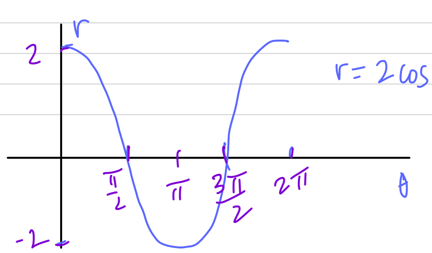
Alternate Integral

$$= 2\left[\frac{1}{2}\int_{0}^{\pi/3} (3\cos\theta)^2 - (1+\cos\theta)^2 \, d\theta\right]$$

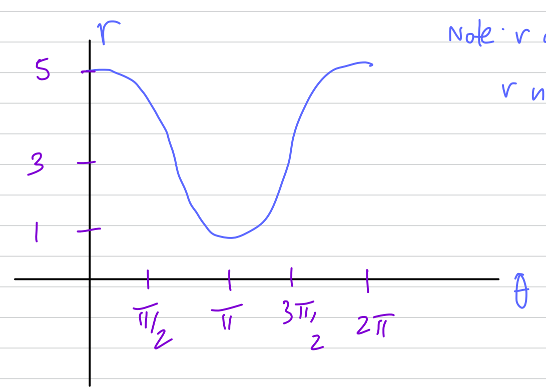
Double using Symmetry

10(d) $r = 3+2\cos\theta$ $\quad$ Cartesian Plots First





Shift up 3 units.



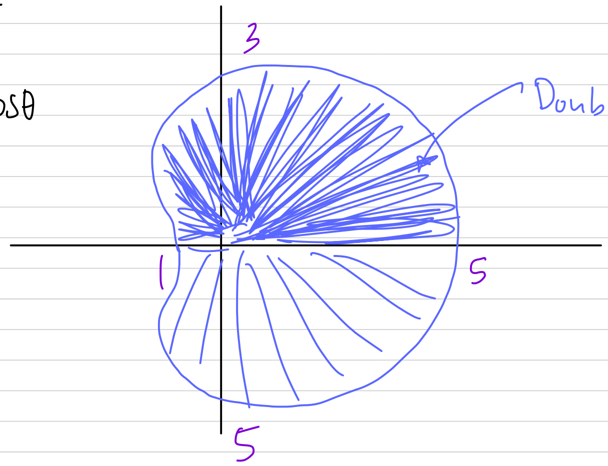
Note: r all (+)

r never 0

r: 5 ↘ 3 ↘ 1 ↗ 3 ↗ 5

Polar Plot

$r = 3 + 2\cos\theta$



$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (\text{Polar Radius})^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (3 + 2\cos\theta)^2 \, d\theta$$

Alternate Integral

$$= 2\left[\frac{1}{2}\int_0^{\pi} (3 + 2\cos\theta)^2 \, d\theta\right]$$

Double by Symmetry