Extra Examples of Trigonometric Integrals

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ODD POWER TECHNIQUE

1.

$$\int \sin^3 x \ \cos^5 x \ dx = \int \sin^3 x \ \cos^4 x \ \cos x \ dx$$ isolate one copy of $\cos x$

$$= \int \sin^3 x \left(\cos^2 x\right)^2 \cos x \ dx$$ find remaining even powers of $\cos x$

$$= \int \sin^3 x \left(1 - \sin^2 x\right)^2 \cos x \ dx$$ convert $\cos^2 x$ using trig. identity

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x \ dx \end{aligned}}$$ don't need to expand algebra yet ...

$$= \int u^3 \left(1 - u^2\right)^2 du$$ after u-substitution

$$= \int u^3 \left(1 - 2u^2 + u^4\right) du$$ expand algebra now

$$= \int u^3 - 2u^5 + u^7 \ du$$ distribute algebra now

$$= \frac{u^4}{4} - \frac{2u^6}{6} + \frac{u^8}{8} + C$$ integrate power rules

$$= \boxed{\frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C}$$ substitute back to original variable

Since both powers of this problem are odd, you could isolate one copy of $\sin x$ instead. Then proceed in a similar way by converting the remaining even powers of $\sin x$ to cosines. That is, there is some flexibility when both powers are odd ...

Contrast this technique with Problem 5 below, where there are both even powers of $\sin x$ and $\cos x$.

What makes this *odd power* technique work, is that when you isolate one of the odd powers of $\sin x$ (or $\cos x$), you are left with an *even* number of $\sin x$ (or $\cos x$). That allows you to use the trig. identity $\sin^2 x + \cos^2 x = 1$, to convert those remaining even powers of $\sin x$ to $\cos x$, (or $\cos x$ to $\sin x$).

2.

$$\int \sin^5 x \, dx = \int \sin^4 x \; \sin x \, dx$$ isolate one copy of $\sin x$

$$= \int \left(\sin^2 x\right)^2 \sin x \, dx$$ find remaining even powers of $\sin x$

$$= \int \left(1 - \cos^2 x\right)^2 \sin x \, dx$$ convert $\sin^2 x$ using trig. identity

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \\ -du &= \sin x \, dx \end{aligned}}$$ don't need to expand algebra yet . . .

$$= -\int \left(1 - u^2\right)^2 \, du$$ after u-substitution

$$= -\int 1 - 2u^2 + u^4 \, du$$ expand algebra now

$$= -u + \frac{2u^3}{3} - \frac{u^5}{5} + C$$ integrate power rules, distribute minus sign

$$= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C}$$ substitute back to original variable

Contrast this technique with Problem 6 below, where there is an even power of $\sin x$.

Some useful Trigonometric identities:

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

EVEN POWER TECHNIQUE: using half-angle trigonometric identities

3.

$$\int \sin^2 x\, dx = \int \frac{1-\cos(2x)}{2}\, dx \quad \text{half-angle identity for } \sin^2 x$$

$$= \frac{1}{2}\int 1-\cos(2x)\, dx \quad \text{factor out to simplify}$$

$$= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C \quad \text{integrate, maybe need } u\text{-substitution}$$

$$= \boxed{\frac{x}{2} - \frac{1}{4}\sin(2x) + C} \quad \text{simplify}$$

4.

$$\int \cos^2 x\, dx = \int \frac{1+\cos(2x)}{2}\, dx \quad \text{half-angle identity for } \cos^2 x$$

$$= \frac{1}{2}\int 1+\cos(2x)\, dx \quad \text{factor out to simplify}$$

$$= \frac{1}{2}\left(x + \frac{\sin(2x)}{2}\right) + C \quad \text{integrate, maybe need } u\text{-substitution}$$

$$= \boxed{\frac{x}{2} + \frac{1}{4}\sin(2x) + C} \quad \text{simplify}$$

5.

$$\int \sin^2 x\ \cos^2 x dx = \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) dx \quad \text{half-angle identities for } \sin^2 x,\ \cos^2 x$$

$$= \frac{1}{4}\int 1-\cos^2(2x)\, dx \quad \text{expand algebra}$$

$$= \frac{1}{4}\int 1-\left(\frac{1+\cos(4x)}{2}\right)\, dx \quad \text{half-angle identity \ldots watch the } 4x$$

$$= \frac{1}{4}\int 1-\frac{1}{2}-\frac{\cos(4x)}{2}\, dx \quad \text{simplify algebra}$$

$$= \frac{1}{8}\int 1-\cos(4x)\, dx \quad \text{factor out}$$

$$= \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C \quad \text{integrate, maybe need } u\text{-substitution}$$

$$= \boxed{\frac{x}{8} - \frac{1}{32}\sin(4x) + C} \quad \text{simplify}$$

6.

$$\int \sin^4 x \; dx = \int \left(\sin^2 x\right)^2 \; dx$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \; dx \qquad \text{half-angle identity for } \sin^2 x$$

$$= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x) \; dx \qquad \text{expand algebra}$$

$$= \frac{1}{4}\int 1 - 2\cos(2x) + \left(\frac{1+\cos(4x)}{2}\right) \; dx \qquad \text{half-angle identity \dots watch the } 4x$$

$$= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2} + \frac{\cos(4x)}{2} \; dx \qquad \text{simplify algebra}$$

$$= \frac{1}{4}\int \frac{3}{2} - 2\cos(2x) + \frac{\cos(4x)}{2} \; dx \qquad \text{simplify algebra again}$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin(2x) + \frac{\sin(4x)}{8}\right) + C \qquad \text{integrate, maybe need } u\text{-substitution}$$

$$= \boxed{\frac{3}{8}x - \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C} \qquad \text{simplify}$$

7.

$$\int \cos^4(3x) \; dx = \int \left(\cos^2(3x)\right)^2 \; dx$$

$$= \int \left(\frac{1+\cos(6x)}{2}\right)^2 \; dx \qquad \text{half-angle identity \dots watch the } 6x$$

$$= \frac{1}{4}\int 1 + 2\cos(6x) + \cos^2(6x) \; dx \qquad \text{expand algebra}$$

$$= \frac{1}{4}\int 1 + 2\cos(6x) + \left(\frac{1+\cos(12x)}{2}\right) \; dx \qquad \text{half-angle identity \dots watch the } 12x$$

$$= \frac{1}{4}\int 1 + 2\cos(6x) + \frac{1}{2} + \frac{\cos(12x)}{2} \; dx \qquad \text{simplify algebra}$$

$$= \frac{1}{4}\int \frac{3}{2} + 2\cos(6x) + \frac{\cos(12x)}{2} \; dx \qquad \text{simplify algebra again}$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \frac{2\sin(6x)}{6} + \frac{\sin(12x)}{24}\right) + C \qquad \text{integrate, maybe need } u\text{-substitution}$$

$$= \boxed{\frac{3}{8}x + \frac{\sin(6x)}{12} + \frac{\sin(12x)}{96} + C} \qquad \text{simplify}$$