Quiz #8 Final Answers

1(a) $$\frac{d}{dx}\left(x^3 e^{-x^5}\right) = \frac{d}{dx}\left(x^3 \sum_{n=0}^{\infty} \frac{(-x^5)^n}{n!}\right) = \frac{d}{dx}\left(x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n}}{n!}\right)$$

$R=\infty$; $(-1)^n x^{5n}$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+3}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (5n+3) x^{5n+2}}{n!}$$

$R=\infty$ STILL

$R=\frac{1}{5}$

Need $|5x|<1 \Rightarrow |x|<\frac{1}{5}$

$R=1$

1(b) $$\int x^3 \ln(1+5x)\,dx = \int x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{n+1}}{n+1}\,dx = \int x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} x^{n+1}}{n+1}\,dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} x^{n+4}}{n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{n+1} x^{n+5}}{(n+1)(n+5)} + C$$

$R=\frac{1}{5}$ STILL

OR, can leave this 3

$R=\infty$

1(c) $$\frac{d}{dx}\left(3x^2\cos(3x)\right) = \frac{d}{dx}\left(3x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!}\right) = \frac{d}{dx}\left(3x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!}\right)$$

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+2}}{(2n)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} (2n+2) x^{2n+1}}{(2n)!}$$

$R=\infty$ STILL

2. Recall $$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos\left(\tfrac{1}{2}\right) = 1 - \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^4}{4!} - \frac{\left(\frac{1}{2}\right)^6}{6!} + \cdots$$

$$= 1 - \frac{1}{8} + \frac{1}{384} - \cdots$$

$$\approx 1 - \frac{1}{8} = \frac{7}{8}$$ ← Estimate

$$\begin{array}{r} 24 \\ \times 16 \\ \hline 144 \\ 240 \\ \hline 384 \end{array}$$

Using ASET, this estimate yields Error at most $\frac{1}{384} < \frac{1}{200}$ as desired

3(a)

$f(x) = \cosh x \qquad f(0) = \cosh 0 = 1$

$f'(x) = \sinh x \qquad f'(0) = \sinh 0 = 0$

$f''(x) = \cosh x \qquad f''(0) = \cosh 0 = 1$

$f'''(x) = \sinh x \qquad f'''(0) = \sinh 0 = 0$

$f^{(4)}(x) = \cosh x \qquad f^{(4)}(0) = \cosh 0 = 1$

$\vdots \qquad \vdots$

MacLaurin Series

$$\cancel{f(0)}^{1} + \underset{0}{\cancel{f'(0)}}x + \frac{\cancel{f''(0)}^{1}}{2!}x^2 + \frac{\underset{0}{\cancel{f'''(0)}}}{3!}x^3 + \frac{\cancel{f^{(4)}(0)}^{1}}{4!}x^4 + \frac{\underset{0}{\cancel{f^{(5)}(0)}}}{5!}x^5 + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4} + \frac{x^6}{6!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Note: Looks like formula for Cosine, but it's not alternating.

3(b) Substitution

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{\overbrace{(-x)^n}^{(-1)^n x^n}}{n!}\right)$$

$$= \frac{1}{2}\left(\left(1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} + \cdots\right) + \left(1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} - \cdots\right)\right)$$

All ODD Power Cancel

$$= \cancel{\frac{1}{2}}\left(\cancel{2} + \cancel{2}\left(\frac{x^2}{2!}\right) + \cancel{2}\left(\frac{x^4}{4!}\right) + \cdots\right)$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Match

For Fun

Alternate Methods: OR, any of these could be for part (b) above.

$\boxed{1}$ Use either chart method OR substitution to find the Power Series for sinhx

↳ then Differentiate OR Integrate

Substitution $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$$= \frac{1}{2}\left(\left(\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cdots\right) - \left(\cancel{1} - x + \cancel{x^2} - \frac{x^3}{3!} + \cdots\right)\right)$$

$$= \cancel{\frac{1}{2}}\left(\cancel{2}x + \cancel{2}\left(\frac{x^3}{3!}\right) + \cancel{2}\left(\frac{x^5}{5!}\right) + \cdots\right)$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Next: $\cosh x = \frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\right)$

$$= \sum_{n=0}^{\infty} \frac{\cancel{(2n+1)}\, x^{2n}}{\underbrace{\cancel{(2n+1)}!}_{(2n)!}} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$ Match

OR Long Form

$$\cosh x = \frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)$$

$$= 1 + \frac{1}{\underbrace{\cancel{3!}}_{2!}}(\cancel{3}x^2) + \frac{1}{\underbrace{\cancel{5!}}_{4!}}(\cancel{5}x^4) + \frac{1}{\underbrace{\cancel{7!}}_{6!}}(\cancel{7}x^6) + \cdots$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$ Match

$\boxed{2}$ Integration of sinhx → Need to find +C

$$\cosh x = \int \sinh x \, dx = \int \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{\underbrace{(2n+1)!\,(2n+2)}_{(2n+2)!}} + C$$

Solve for +C

$$\cosh x = \underbrace{\frac{x^2}{2!}}_{n=0} + \underbrace{\frac{x^4}{4!}}_{n=1} + \underbrace{\frac{x^6}{6!}}_{n=2} + \cdots + \cancel{C} \to 1$$

Test $x=0$

$$\cancel{\cosh 0}^{\,1} = 0 + \cancel{0 + 0 + \cdots}^{\,0} + C \Rightarrow C = 0$$

Collect terms

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$