• Please see the course webpage for the answer key.

1. Find the Interval and Radius of Convergence for the folllowing power series. Analyze carefully and with full justification.

$$\sum_{n=0}^{\infty} \frac{(-1)^n (7x-3)^n}{(n+1)\ 5^{n+1}}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(7x-3)^{n+1}}{(n+2)5^{n+2}}}{\frac{(-1)^n(7x-3)^n}{(n+1)5^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(7x-3)^{n+1}}{(7x-3)^n} \right| \cdot \left(\frac{n+1}{n+2} \right) \cdot \frac{5^{n+1}}{5^{n+2}} = \frac{|7x-3|}{5}$$

The Ratio Test gives convergence for x when $\frac{|7x-3|}{5} < 1$ or $|7x-3| < 5$.

That is $-5 < 7x - 3 < 5 \Longrightarrow -2 < 7x < 8 \Longrightarrow -\frac{2}{7} < x < \frac{8}{7}$

Endpoints:

•$x = \frac{8}{7}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(7\left(\frac{8}{7}\right) - 3\right)^n}{(n+1)\ 5^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{(n+1)\ 5^{n+1}} = \frac{1}{5}\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

which is **convergent** because it's a constant multiple of a series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ which is itself convergent by AST:

1. $b_n = \frac{1}{n+1} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$

3. $b_{n+1} < b_n$ because $b_{n+1} = \frac{1}{n+2} < \frac{1}{n+1} = b_n$.

OR $f(x) = \frac{1}{x+1}$ has derivative $f'(x) = -\frac{1}{(x+1)^2} < 0$ so the terms are decreasing.

•$x = -\frac{2}{7}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(7\left(-\frac{2}{7}\right) - 3\right)^n}{(n+1)\ 5^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n(-5)^n}{(n+1)\ 5^{n+1}}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n 5^n}{(n+1)5^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{5(n+1)} = \frac{1}{5}\sum_{n=0}^{\infty} \frac{1}{n+1}$$

Here

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ the divergent Harmonic Series, $p = 1$.

LCT: $\lim_{n\to\infty} \dfrac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \dfrac{n}{n+1} = 1$ which is *finite* and *non-zero*. Therefore, $\sum_{n=0}^{\infty} \dfrac{1}{n+1}$ is also divergent by LCT.

Also, the series $\dfrac{1}{5}\sum_{n=0}^{\infty} \dfrac{1}{n+1}$ is **divergent** as a constant multiple of a divergent series above.

Finally, Interval of Convergence $\boxed{I = \left(-\dfrac{2}{7}, \dfrac{8}{7}\right]}$ with Radius of Convergence $\boxed{R = \dfrac{5}{7}}$.

2. Find the Interval and Radius of Convergence for the folllowing power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ \ln n \ (x+5)^n}{n^2 \ 4^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1} \ \ln(n+1) \ (x+5)^{n+1}}{(n+1)^2 4^{n+1}}}{\dfrac{(-1)^n \ \ln n \ (x+5)^n}{n^2 4^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{(x+5)^n} \right| \cdot \left(\frac{n}{n+1}\right)^2 \cdot \left(\frac{\ln(n+1)}{\ln(n)}\right) \cdot \frac{4^n}{4^{n+1}} \overset{(*)\ \text{L'H}}{=} \frac{|x+5|}{5} \text{ (see below)}$$

The Ratio Test gives convergence for x when $\dfrac{|x+5|}{4} < 1$ or $|x+5| < 4$.

That is $-4 < x+5 < 4 \Longrightarrow -9 < x < -1$

Endpoints:

•$x = -9$ The original series becomes $\sum_{n=1}^{\infty} \dfrac{(-1)^n \ln n \ (-4)^n}{n^2 \ 4^n} = \sum_{n=1}^{\infty} \dfrac{(-1)^n \ln n \ (-1)^n 4^n}{n^2 \ 4^n} = \sum_{n=1}^{\infty} \dfrac{\ln n}{n^2}$

We use the bound (as n gets large) $\ln n \le \sqrt{n}$ and bound the terms $\dfrac{\ln n}{n^2} < \dfrac{\sqrt{n}}{n^2} = \dfrac{1}{n^{\frac{3}{2}}}$.

We know $\sum_{n=1}^{\infty} \dfrac{1}{n^{\frac{3}{2}}}$ is a convergent p-series with $p = \dfrac{3}{2} > 1$. The original (smaller) series is **convergent** by CT.

Endpoints:

•$x = -1$ The original series becomes $\sum_{n=1}^{\infty} \dfrac{(-1)^n \ln n \ (4)^n}{n^2 \ 4^n} = \sum_{n=1}^{\infty} \dfrac{(-1)^n \ln n}{n^2}$

Consider the absolute series $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ which was shown convergent above. Therefore, this orginal series **converges** by ACT.

Finally, Interval of Convergence $\boxed{I = [-9, -1]}$ with Radius of Convergence $\boxed{R = 4}$.

$$(*)\ \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}=\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x}\overset{\frac{\infty}{\infty}\ \text{L'H}}{=}\lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{x}{x+1}\overset{\frac{\infty}{\infty}\ \text{L'H}}{=}\lim_{x\to\infty}\frac{1}{1}=1$$

3. Find the MacLaurin Series representation for each of the following functions. State the Radius of Convergence for each series. You answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(a) $f(x)=\dfrac{x^2}{1+5x}=\dfrac{x^2}{1-(-5x)}=x^2\sum_{n=0}^{\infty}(-5x)^n=x^2\sum_{n=0}^{\infty}(-1)^n5^nx^n=\boxed{\sum_{n=0}^{\infty}(-1)^n5^nx^{n+2}}$

Here need $|-5x|<1$ or $|x|<\frac{1}{5}$, so $\boxed{R=\frac{1}{5}}$.

(b) $f(x)=x^7\sin(x^2)$

First, $\sin x=\sum_{n=0}^{\infty}\dfrac{(-1)^nx^{2n+1}}{(2n+1)!}$. Here $R=\infty$.

Then, $\sin(x^2)=\sum_{n=0}^{\infty}\dfrac{(-1)^n(x^2)^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}\dfrac{(-1)^nx^{4n+2}}{(2n+1)!}$. $R=\infty$ still.

Finally, $x^7\sin(x^2)=x^7\sum_{n=0}^{\infty}\dfrac{(-1)^nx^{4n+2}}{(2n+1)!}=\boxed{\sum_{n=0}^{\infty}\dfrac{(-1)^nx^{4n+9}}{(2n+1)!}}$ $\boxed{R=\infty}$ still.

(c) $f(x)=x\arctan(3x)$

First, $\arctan x=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$.

Next, $\arctan(3x)=\sum_{n=0}^{\infty}(-1)^n\dfrac{(3x)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\dfrac{3^{2n+1}x^{2n+1}}{2n+1}$

Finally, $x\arctan(3x)=x\sum_{n=0}^{\infty}(-1)^n\dfrac{3^{2n+1}x^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\dfrac{3^{2n+1}x^{2n+2}}{2n+1}$

Here need $|3x|<1$ or $|x|<\frac{1}{3}$, so $\boxed{R=\frac{1}{3}}$.

(d) $f(x)=x^4e^{-x^3}$

First, $e^x=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}$ Here $R=\infty$.

Then, $e^{-x^3}=\sum_{n=0}^{\infty}\dfrac{(-x^3)^n}{n!}=\sum_{n=0}^{\infty}\dfrac{(-1)^nx^{3n}}{n!}$. $R=\infty$ still.

Finally, $x^4 e^{-x^3} = x^4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+4}}{n!}}$. $\boxed{R = \infty}$ still.

(e) $f(x) = x^3 \ln(1 + x^3)$

First $\ln(1 + x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n + 1}$ Here $R = 1$.

Second, $\ln(1 + x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{n+1}}{n + 1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n + 1}$ $R = 1$ still.

Finally, $x^3 \ln(1 + x^3) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{n + 1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{n + 1}}$ $\boxed{R = 1}$ still.

(f) $f(x) = x^2 \cos(4x)$

First, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Here $R = \infty$.

Then, $\cos(4x) = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!}$. $R = \infty$ still.

Finally, $x^2 \cos(4x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n} x^{2n+2}}{(2n)!}}$ $\boxed{R = \infty}$ still.