Math 121 Extra Partial Fraction Examples

1. Compute $\displaystyle\int \frac{x^2+7}{x^2+2x+10}\,dx$

Note that the integrand is an improper rational function because the degree of the numberator is *not* strictly less than that of the denominator. We apply long division of polynomials to write the integrand as a (simple) polynomial and a proper rational piece. That is, the degree of the numerator is strictly less than that of the denominator.

Long division yields:

$$x^2+2x+10\overline{\smash{)}x^2+7}^{\;1}$$

$$\underline{-(x^2+2x+10)}$$

$$-2x-3$$

That means $x^2+7=(x^2+2x+10)(1)+(-2x-3)$. Here the remainder term is $-2x-3$. Dividing both sides by $x^2+2x+10$ yields $\displaystyle\frac{x^2+7}{x^2+2x+10}=1+(-2x-3)$

Now $\displaystyle\int \frac{x^2+7}{x^2+2x+10}\,dx=\int 1+\frac{-2x-3}{x^2+2x+10}\,dx=\int 1-\frac{2x+3}{x^2+2x+10}\,dx$

The second piece contains a quadratic irreducible, so partial fractions will not be helpful here. *Complete the square* on that irreducible piece to convert it into an "almost" arctan integral.

$=\displaystyle\int 1-\frac{2x+3}{(x+1)^2+9}\,dx=\int 1\,dx-\int\frac{2(u-1)+3}{u^2+9}\,du$ • see subst. below

$=\displaystyle\int 1\,dx-\int\frac{2u-2+3}{u^2+9}\,du=\int 1\,dx-\int\frac{2u+1}{u^2+9}\,du$ • simplify

$=\displaystyle\int 1\,dx-\int\frac{2u}{u^2+9}\,du-\int\frac{1}{u^2+9}\,du$ • after split of integrals

• here we have a simple term, a natural log term, and an "arctan"-ish term

• make sure that you understand how to compute the last two integrals quickly

$$=x-\ln|u^2+9|-\frac{1}{3}\arctan\left(\frac{u}{3}\right)+C=\boxed{x-\ln|(x+1)^2+9|-\frac{1}{3}\arctan\left(\frac{x+1}{3}\right)+C}$$

We ("invertedly") substituted above $\boxed{\begin{array}{l}u=x+1\Rightarrow x=u-1\\ du=dx\end{array}}$

2. Compute $\int \frac{x+13}{x(x^2+4x+13)}\,dx$

Note that the integrand is already a proper rational function. The denominator is already factored into a linear factor and a quadratic irreducible factor. (why is that irreducible?) We use the following Partial Fractions decomposition:

$$\frac{x+13}{x(x^2+4x+13)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+13}$$

Clearing the denominator yields:

$x+13 = A(x^2+4x+13) + (Bx+C)x$
$x+13 = Ax^2+4Ax+13A+Bx^2+Cx$
$x+13 = (A+B)x^2+(4A+C)x+13A$
so that $\boxed{A+B=0}$ and $\boxed{4A+C=1}$ and $\boxed{13A=13}$

Solve for $\boxed{A=1}$ and $\boxed{B=-1}$ and $\boxed{C=-3}$

Finally, we have decomposed the original integrand. Hopefully this new, but equal, decomposition is *easier* to integrate.

$$\frac{x+13}{x(x^2+4x+13)} = \frac{1}{x} + \frac{-x-3}{x^2+4x+13}$$

Now,

$$= \int \frac{x+13}{x(x^2+4x+13)}\,dx = \int \frac{1}{x} + \frac{-x-3}{x^2+4x+13}\,dx = \int \frac{1}{x} - \frac{x+3}{x^2+4x+13}\,dx$$

- complete the square on the quadratic irreducible second term

$$= \int \frac{1}{x} - \frac{x+3}{(x+2)^2+9}\,dx = \int \frac{1}{x}\,dx - \int \frac{(u-2)+3}{u^2+9}\,du \quad \bullet \text{ see subst. below}$$

$$= \int \frac{1}{x}\,dx - \int \frac{u+1}{u^2+9}\,du = \int \frac{1}{x}\,dx - \int \frac{u}{u^2+9}\,du - \int \frac{1}{u^2+9}\,du \quad \bullet \text{ after split of integrals}$$

- here we have some natural log terms, and an "arctan"-ish term

- make sure that you understand how to compute the last two integrals quickly

$$= \ln|x| - \frac{1}{2}\ln|u^2+9| - \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \boxed{\ln|x| - \frac{1}{2}\ln|(x+2)^2+9| - \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C}$$

We ("invertedly") substituted above $\boxed{\begin{array}{l} u = x+2 \Rightarrow x = u-2 \\ du = dx \end{array}}$

3. Compute $\displaystyle\int \frac{1}{(x-2)(x^2+1)}\,dx$

Note that the integrand is a proper rational function because the degree of the numberator is strictly less than that of the denominator.

We use the following Partial Fractions decomposition:

$$\frac{1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

Clearing the denominator yields:

$1 = A(x^2+1) + (Bx+C)(x-2)$
$1 = Ax^2 + A + Bx^2 + Cx - 2Bx - 2C$
$1 = (A+B)x^2 + (C-2B)x + A - 2C$
so that $\boxed{A+B=0}$ and $\boxed{C-2B=0}$ and $\boxed{A-2C=1}$

Solve for $\boxed{A=\frac{1}{5}}$ and $\boxed{B=-\frac{1}{5}}$ and $\boxed{C=-\frac{2}{5}}$

Finally, we have decomposed the original integrand. Hopefully this new, but equal, decomposition is *easier* to integrate.

$$\int \frac{1}{(x-2)(x^2+1)}\,dx = \int \frac{\frac{1}{5}}{x-2} + \frac{-\frac{1}{5}x - \frac{2}{5}}{x^2+1}\,dx$$ • how to handle?

$$= \frac{1}{5}\int \frac{1}{x-2}\,dx - \frac{1}{5}\int \frac{x}{x^2+1}\,dx - \frac{2}{5}\int \frac{1}{x^2+1}\,dx$$

• after split of integrals
• here we have some natural log terms (how?), and an "arctan"-ish term
• make sure that you understand how to compute these integrals quickly

$$= \frac{1}{5}\ln|x-2| - \frac{1}{5}\frac{\ln|x^2+1|}{2} - \frac{2}{5}\arctan x + C$$

$$= \boxed{\frac{1}{5}\ln|x-2| - \frac{1}{10}\ln|x^2+1| - \frac{2}{5}\arctan x + C}$$

4. Compute $\displaystyle\int \frac{1}{(x-1)(x^2+x+1)}\,dx$

Note that the integrand is a proper rational function because the degree of the numberator is strictly less than that of the denominator. We will *Complete the square* on the irreducible factor in order to convert it into an "almost" arctan integral.

For now, we use the following Partial Fractions decomposition:

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Clearing the denominator yields:

$1 = A(x^2+x+1) + (Bx+C)(x-1)$
$1 = Ax^2 + Ax + A + Bx^2 + Cx - Bx - C$
$1 = (A+B)x^2 + (A-B+C)x + A - C$
so that $\boxed{A+B=0}$ and $\boxed{A-B+C=0}$ and $\boxed{A-C=1}$

Solve for $\boxed{A=\frac{1}{3}}$ and $\boxed{B=-\frac{1}{3}}$ and $\boxed{C=-\frac{2}{3}}$

Finally, we have decomposed the original integrand. Hopefully this new, but equal, decomposition is *easier* to integrate.

$$\int \frac{1}{(x-1)(x^2+x+1)}\,dx = \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx$$

$$= \int \frac{\frac{1}{3}}{x-1}\,dx - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

• complete the square on the quadratic irreducible second term

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{\left(u-\frac{1}{2}\right)+2}{u^2+\frac{3}{4}}\,du$$ • see subst. below

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{u+\frac{3}{2}}{u^2+\frac{3}{4}}\,du$$ • simplify

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du - \frac{1}{3}\int \frac{\frac{3}{2}}{u^2+\frac{3}{4}}\,du$$

• after split of integrals
• here we have some natural log terms (how?), and an "arctan"-ish term

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du - \frac{1}{3}\int \frac{\frac{3}{2}}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$$

• make sure that you understand how to compute these integrals quickly

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\frac{\ln|u^2+\frac{3}{4}|}{2} - \frac{1}{3}\left(\frac{3}{2}\right)\left(\frac{2}{\sqrt{3}}\right)\arctan\left(\frac{u}{\left(\frac{\sqrt{3}}{2}\right)}\right) + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|u^2+\frac{3}{4}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right) + C$$

$$= \boxed{\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

We ("invertedly") substituted above $\boxed{\begin{array}{l} u = x+\frac{1}{2} \Rightarrow x = u-\frac{1}{2} \\ du = dx \end{array}}$