Exam #3 Spring 2020

(Taken Remotely)

1.(a). $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (5x+1)^n}{(n+1)\,9^n}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(5x+1)^{n+1}}{(n+2)\,9^{n+1}}}{\frac{\cancel{(-1)^n}(5x+1)^n}{(n+1)\,9^n}}\right| = \lim_{n\to\infty}\left|\frac{(5x+1)^{n+1}}{(5x+1)^n}\right|\cdot\left(\frac{n+1}{n+2}\right)\frac{9^n}{9^{n+1}}$$

$|5x+1|$; $\frac{n+1}{n+2} \to 1$; $\frac{9^n}{9^{n+1}} \to 1/9$

no justify needed for Exam #3

$$= \frac{|5x+1|}{9} < 1$$

Converges by R.T. when

$|5x+1| < 9 \Rightarrow -9 < 5x+1 < 9$

$-10 < 5x < 8$

$-2 < x < 8/5$

$10/5 \quad 2 \qquad 8/5$

$-2 \qquad 0 \qquad 8/5$

$18/5 \to$ Half $9/5$

Manually Test Convergence at Endpoints

Take $x = 8/5$ O.S. Becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n\left[5(8/5)+1\right]^n}{(n+1)\,9^n} = \sum_{n=1}^{\infty}\frac{(-1)^n \cancel{9^n}}{(n+1)\cancel{9^n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$$

Converges by AST

① $b_n = \frac{1}{n+1} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n+1} = 0$

③ $b_{n+1} = \frac{1}{n+2} \le \frac{1}{n+1} = b_n$

Terms decreasing ✓

Take $x = -2$ O.S. Becomes

EVEN

$(-1)^{2n} \to 1$

$(-1)^n 9^n$

$$\sum_{n=1}^{\infty} \frac{(-1)^n\left[5(-2)+1\right]^n}{(n+1)\,9^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-9)^n}{(n+1)\,9^n} = \sum_{n=1}^{\infty}\frac{1}{n+1} \sim \sum_{n=1}^{\infty}\frac{1}{n}$$

Diverges p-Series $p=1$

LCT

$$\lim_{n\to\infty}\frac{\frac{1}{n+1}}{1/n} = \lim_{n\to\infty}\frac{n}{n+1} = 1 \text{ Finite, Non-Zero}$$

$\Rightarrow$ Diverges by LCT

Finally,

$$\boxed{\begin{aligned} I &= (-2, 8/5] \\ R &= 9/5 \end{aligned}}$$

1(b) $\sum_{n=1}^{\infty} n^n (x-3)^n$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}(x-3)^{n+1}}{n^n (x-3)^n}\right| = \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{\to e}\cdot \underbrace{(n+1)}_{\to\infty}\,|x-3| = \infty > 1$$

Diverges by R.T. unless $x=3$.

$$\boxed{\begin{array}{l} I = \{3\} \\ R = 0 \end{array}}$$

3 only

1(c) $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n!}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{2(n+1)+1}}{(n+1)!}}{\frac{x^{2n+1}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{x^{2n+1}}\cdot\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\frac{|x|^2}{n+1} = 0 < 1$$

$(n+1)! = (n+1)n!$

Fixed

always

$n+1 \to \infty$

Converges by R.T. for all x in $\mathbb{R}$.

$$\boxed{\begin{array}{l} I = (-\infty, \infty) \\ R = \infty \end{array}}$$

2(a). $$\frac{x^4}{1+7x} = x^4\left[\frac{1}{1+7x}\right] = x^4\left[\frac{1}{1-(-7x)}\right] = x^4\sum_{n=0}^{\infty}(-7x)^n$$

$$= x^4\sum_{n=0}^{\infty}(-1)^n 7^n x^n = \boxed{\sum_{n=0}^{\infty}(-1)^n 7^n x^{n+4}}$$

Need $|-7x| < 1 \Rightarrow |x| < 1/7 \Rightarrow \boxed{R = 1/7}$

$-1/7 < x < 1/7$

$-1/7$ 0 $1/7$

$$2(b)\quad x^3\sin(x^2) = x^3\sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = x^3\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+5}}{(2n+1)!}}$$

$\boxed{R=\infty}$

$R=\infty$

$$2(c)\quad \int x^3\sin(x^2)\,dx = \int\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+5}}{(2n+1)!}\,dx = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{x^{4n+6}}{(4n+6)} + C}$$

$$2(d)\quad x^2\ln(1+5x) = x^2\sum_{n=0}^{\infty}\frac{(-1)^n(5x)^{n+1}}{n+1} = x^2\sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}x^{n+1}}{n+1} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}x^{n+3}}{n+1}}$$

Need $|5x|<1 \Rightarrow |x|<1/5$ $\boxed{R=\frac{1}{5}}$

$-1/5 < x < 1/5$

$$2(e)\quad \frac{d}{dx}\left[x^2\ln(1+5x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}x^{n+3}}{n+1}\right] = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 5^{n+1}(n+3)x^{n+2}}{n+1}}$$

$$3(a)\quad \frac{1}{\sqrt{e}} = e^{-1/2} = 1+(-1/2)+\frac{(-1/2)^2}{2!}+\frac{(-1/2)^3}{3!}+\frac{(-1/2)^4}{4!}+\dots = 1-\frac{1}{2}+\frac{(1/4)}{2!}-\frac{\left(\frac{1}{8}\right)}{3!}+\frac{\left(\frac{1}{16}\right)}{4!}-\dots$$

$4! \leftarrow 24$

$$= 1-1/2+1/8-\frac{1}{48}+\frac{1}{384}-\dots$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$$

$$\approx 1-1/2+1/8-1/48$$

$$= \frac{48}{48}-\frac{24}{48}+\frac{6}{48}-\frac{1}{48} = \boxed{\frac{29}{48}}$$ Estimate

$$\begin{array}{r} 24 \\ \times\ 16 \\ \hline 144 \\ 240 \\ \hline 384 \end{array}$$

Using ASET, the error will be at most $\frac{1}{384}<\frac{1}{200}$ as desired.

$$3(b)\quad \sin(1) = 1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\dots = 1-\frac{1}{6}+\frac{1}{120}-\frac{1}{5040}+\dots$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$$

$$\approx 1-\frac{1}{6}+\frac{1}{120} = \frac{120}{120}-\frac{20}{120}+\frac{1}{120} = \boxed{\frac{101}{120}}$$ ← Estimate

Using ASET, the error is at most $\frac{1}{5040}<\frac{1}{1000}$ as desired.

$$4(a)\ \sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{6^{2n}(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!}\cdot\frac{\left(\frac{\pi}{6}\right)}{\boxed{\left(\frac{\pi}{6}\right)}}=\frac{6}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\pi/6\right)^{2n+1}}{(2n+1)!}$$

$$=\frac{6}{\pi}\,\cancelto{1/2}{\sin\left(\frac{\pi}{6}\right)}=\boxed{\frac{3}{\pi}}$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$4(b)\ \sum_{n=0}^{\infty}\frac{(-1)^n(\ln 9)^n}{2^n\cdot n!}=\sum_{n=0}^{\infty}\frac{\left(\frac{-\ln 9}{2}\right)^n}{n!}=e^{\frac{-\ln 9}{2}}=e^{-\frac{1}{2}\cdot\ln 9}=\cancel{e^{\ln}}\left[9^{-1/2}\right]=\frac{1}{\sqrt{9}}=\boxed{\frac{1}{3}}$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$4(c)\ 2-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\cdots=2\left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right]=2\ln(1+1)=\boxed{2\ln 2}$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

$$4(d)\ -\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\left[\cancelto{\pi/4}{\arctan 1}\right]-1=\boxed{\frac{\pi}{4}-1}\ \overset{\text{OR}}{=}\ \boxed{\frac{\pi-4}{4}}$$

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\qquad \arctan 1=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

$$4(e)\ \sum_{n=0}^{\infty}\frac{1}{e^n}=\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n=\frac{1}{1-1/e}=\frac{1}{\left(\frac{e-1}{e}\right)}=\boxed{\frac{e}{e-1}}$$

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$$

OR// $\sum_{n=0}^{\infty}\frac{1}{e^n}=\sum_{n=0}^{\infty}e^{-n}=\sum_{n=0}^{\infty}(e^{-1})^n=\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^n=\cdots$ same

$$4(f)\ \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n}}{2^{4n}(2n)!}=-\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2^2)^{2n}(2n)!}=-\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{4^{2n}(2n)!}=-\sum_{n=0}^{\infty}\frac{(-1)^n\left(\pi/4\right)^{2n}}{(2n)!}$$

$$=-\cancelto{+\sqrt{2}/2}{\cos\left(\frac{\pi}{4}\right)}=\boxed{\frac{-\sqrt{2}}{2}}$$

$$5(a)\quad F(x) = \ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \cancelto{0}{+C}$$

What is C?

Test $x=0$. So Far

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + C$$

$$\cancelto{0}{\ln(1+0)} = \cancelto{0}{0 - 0 + 0 - 0 + \cdots} + C \Rightarrow C = 0$$

$$\text{Finally, } \ln(1+x) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}}$$

5(b) Chart Method

$F(x) = \ln(1+x)$	$F(0) = \ln 1 = 0$
$F'(x) = \frac{1}{1+x} = (1+x)^{-1}$	$F'(0) = 1$
$F''(x) = -(1+x)^{-2}$	$F''(0) = -1$
$F'''(x) = 2(1+x)^{-3}$	$F'''(0) = 2$
$F^{(4)}(x) = -6(1+x)^{-4}$	$F^{(4)}(0) = -6$
$F^{(5)}(x) = 24(1+x)^{-5}$	$F^{(5)}(0) = 24$
$\vdots$	$\vdots$

MacLaurin Series

$$\cancelto{0}{F(0)} + \cancelto{1}{F'(0)}x + \frac{\cancelto{-1}{F''(0)}}{2!}x^2 + \frac{\cancelto{2}{F'''(0)}}{3!}x^3 + \frac{\cancelto{-6 \,\leftarrow 3!}{F^{(4)}(0)}}{4!}x^4 + \frac{\cancelto{24 \,\leftarrow 4!}{F^{(5)}(0)}}{5!}x^5 + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}} \quad \text{Match}$$

Chart Method

5(c) $G(x) = \sinh x \qquad G(0) = \sinh 0 = 0$

$G'(x) = \cosh x \qquad G'(0) = \cosh 0 = 1$

$G''(x) = \sinh x \qquad G''(0) = \sinh 0 = 0$

$G'''(x) = \cosh x \qquad G'''(0) = \cosh 0 = 1$

$G^{(4)}(x) = \sinh x \qquad G^{(4)}(0) = \sinh 0 = 0$

$\vdots \qquad \vdots$

Maclaurin Series

$$\cancelto{0}{G(0)} + \cancelto{1}{G'(0)}x + \frac{\cancelto{0}{G''(0)}x^2}{2!} + \frac{\cancelto{1}{G'''(0)}x^3}{3!} + \frac{\cancelto{0}{G^{(4)}(0)}x^4}{4!} + \cdots$$ All Even Powers Dissolve $\to 0$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}}$$

5(d) $$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[e^x - e^{-x}\right] = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right]$$

$$= \frac{1}{2}\left[\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} + \cdots - \left(\cancel{1} - x + \cancel{\frac{x^2}{2!}} - \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} - \cdots\right)\right]$$

$$= \frac{1}{2}\left[2x + 2\left(\frac{x^3}{3!}\right) + 2\left(\frac{x^5}{5!}\right) + 2\left(\frac{x^7}{7!}\right) + \cdots\right]$$ Now 2's Cancel..

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$$ Match

6. $x = \dfrac{e^{2t}}{2} - \dfrac{t^3}{3}$ $\qquad$ $y = 2te^t - 2e^t$

$\dfrac{dx}{dx} = e^{2t} - t^2$ $\qquad$ $\dfrac{dy}{dt} = 2te^t + \cancel{2e^t} - \cancel{2e^t} = 2te^t$

(a) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2te^t}{e^{2t} - t^2}$

$$\left.\frac{dy}{dx}\right|_{t=1} = \boxed{\frac{2e}{e^2 - 1}}$$

(b) $L = \displaystyle\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(e^{2t} - t^2)^2 + (2te^t)^2}\, dt$

$$= \int_0^1 \sqrt{e^{4t} \underbrace{- 2t^2e^{2t}} + t^4 + \underbrace{4t^2e^{2t}}}\, dt = \int_0 \sqrt{e^{4t} + 2t^2e^{2t} + t^4}\, dt$$

$$= \int_0^1 \sqrt{(e^{2t} + t^2)^{\cancel{2}}}\, dt = \int_0^1 e^{2t} + t^2\, dt = \left.\frac{e^{2t}}{2} + \frac{t^3}{3}\right|_0^1$$

$$= \frac{e^2}{2} + \frac{1}{3} - \left(\frac{\cancel{e^0}^{\,1}}{2} + 0\right) = \frac{e^2}{2} + \underset{2/6}{\frac{1}{3}} - \underset{-3/6}{\frac{1}{2}} = \boxed{\frac{e^2}{2} - \frac{1}{6}}$$

Bonus

Some power Series

OR Assume $\ln(1+x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

d/dx $\qquad$ d/dx

$\frac{1}{1+x}$ $\qquad$ $c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$

$\sum_{n=0}^{\infty} (-1)^n x^n$

$1 - x + x^2 - x^3 + x^4 - \cdots$

Equate

$c_1 = 1$

$2c_2 = -1 \Rightarrow c_2 = -1/2$

$3c_3 = 1 \Rightarrow c_3 = 1/3$

$4c_4 = -1 \Rightarrow c_4 = -1/4$

$\vdots$

Bonus: Assume $\ln(1+x) = \sum_{n=0}^{\infty} c_n x^n = \cancel{c_0}^{0} + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$

Note c_0 must be 0 since $x=0$ yields $\cancel{\ln(1+0)}^{0} = c_0 + 0 + 0 + 0 + 0 \cdots$

$\Rightarrow c_0 = 0.$

IBP gives $\ln(1+x) = \frac{d}{dx}\left[x\ln(1+x) - x + \ln(1+x)\right]$

$$\cancel{c_0}^{0} + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots = \frac{d}{dx}\left[x(c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots) - x + (c_1 x + c_2 x^2 + c_3 x^3 + \cdots)\right]$$

$$= \frac{d}{dx}\left[c_1 x^2 + c_2 x^3 + c_3 x^4 + c_4 x^5 + \cdots - x + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\right]$$

$$= \frac{d}{dx}\left[(c_1 - 1)x + (c_1 + c_2)x^2 + (c_2 + c_3)x^3 + (c_3 + c_4)x^4 + \cdots\right]$$

$$= (c_1 - 1) + 2(c_1 + c_2)x + 3(c_2 + c_3)x^2 + 4(c_3 + c_4)x^3 + \cdots$$

Equate Coefficients:

$c_1 - 1 = 0 \Rightarrow c_1 = 1$

$2(c_1 + c_2) = c_1 \quad \cancel{c_1}^{1} + 2c_2 = 0 \Rightarrow c_2 = -\frac{1}{2}$

$3(c_2 + c_3) = c_2 \Rightarrow 2\underset{-1}{\cancel{c_2}} + 3c_3 = 0 \Rightarrow c_3 = \frac{1}{3}$

$4(c_3 + c_4) = c_3 \Rightarrow 3\underset{1}{\cancel{c_3}} + 4c_4 = 0 \Rightarrow c_4 = -\frac{1}{4}$

$$\Rightarrow c_n = \pm\frac{1}{n} \Rightarrow \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \text{ or } \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

You can show $\int \ln(1+x)\,dx \overset{IBP}{=} x\ln(1+x) - x + \ln(1+x)$

Bonus

OR

$$\sum_{n=0}^{\infty} c_n x^n = \frac{d}{dx}\left[x\sum_{n=0}^{\infty} c_n x^n - x + \sum_{n=0}^{\infty} c_n x^n\right]$$

$$\sum^{\infty} c_n x^{n+1}$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^{n+1} - x + \sum_{n=0}^{\infty} c_n x^n\right]$$

$$= \sum_{n=0}^{\infty} c_n (n+1) x^n - 1 + \sum_{n=0}^{\infty} n c_n x^{n-1}$$

split

Cancel

$$= \sum_{n=0}^{\infty} c_n n x^n + \cancel{\sum_{n=0}^{\infty} c_n x^n} - 1 + \sum_{n=0}^{\infty} n c_n x^{n-1}$$

$$= 0 + c_1 x^1 + 2c_2 x^2 + 3c_3 x^3 + 4c_4 x^4 + \cdots - 1$$

$$+ 0 + c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

$$0 = (c_1 - 1) + (c_1 + 2c_2)x + (2c_2 + 3c_3)x^2 + (3c_3 + 4c_4)x^3 + \cdots$$

$c_1 - 1 = 0 \Rightarrow c_1 = 1$

$c_1 + 2c_2 = 0 \Rightarrow c_2 = -\frac{c_1}{2} = -\frac{1}{2}$

$2c_2 + 3c_3 = 0 \Rightarrow c_3 = -\frac{2c_2}{3} = -\frac{2(-\frac{1}{2})}{3} = +\frac{1}{3}$

$\vdots$