Answer Key

Math 121 Midterm Exam #2 April 8-11, 2021

Due Sunday, April 11, in Gradescope by 11:59 pm ET

- This is an *Open Notes* Exam. You can use materials, homeworks problems, lecture notes, etc. that you manually worked on.
- There is **NO** *Open Internet* allowed. You can only access our Main Course Webpage.
- You are not allowed to work on or discuss these problems with anyone, including the Professor or Math Fellow TA.
- Submit your final work in Gradescope in the Exam 2 entry.
- Please *show* all of your work and *justify* all of your answers. No Calculators.

Problem	Score	Possible Points
1		25
2		8
3		15
4		26
5		6
6		3
7		3
8		7
9		7
Total		100

1. [25 Points] Compute the following **Improper** integrals. Simplify all answers. Justify your work.

(a) $\int_e^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_e^t \ln x\cdot x^{-3}\,dx \overset{IBP}{=} \lim_{t\to\infty} \left.\frac{-\ln x}{2x^2}\right|_e^t + \frac{1}{2}\int_e^t \frac{1}{x^3}\,dx$ $\quad \to x^{-3}$

$u = \ln x \qquad dv = x^{-3}\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{x^{-2}}{-2} = \frac{-1}{2x^2}$

$$= \lim_{t\to\infty} \left.\frac{-\ln x}{2x^2}\right|_e^t + \frac{1}{2}\left.\left(\frac{x^{-2}}{-2}\right)\right|_e^t$$

$$= \lim_{t\to\infty} \left.\frac{-\ln x}{2x^2}\right|_e^t - \left.\frac{1}{4x^2}\right|_e^t$$

$$= \lim_{t\to\infty} \underbrace{\frac{-\ln t}{2t^2}}_{0}{}^{\frac{-\infty}{\infty}} + \frac{\cancel{\ln e}^{\,1}}{2e^2} - \underbrace{\frac{1}{4t^2}}_{0} + \frac{1}{4e^2}$$

See (*)

$$= \frac{1}{2e^2} + \frac{1}{4e^2} = \boxed{\frac{3}{4e^2}}$$

(*) $\lim_{t\to\infty}\frac{\ln t}{t^2} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{t\to\infty}\frac{\frac{1}{t}}{2t} \; \frac{1}{2t} = \lim_{t\to\infty}\frac{1}{2t^2} = 0$

1. (Continued) Compute the following **Improper** integrals. Simplify all answers. Justify your work.

(b) $\displaystyle\int_{-2}^{5} \frac{8}{x^2-4x-12}\,dx = \int_{-2}^{5} \frac{8}{(x-6)(x+2)}\,dx = \lim_{t\to -2^+} \int_t^5 \frac{8}{(x-6)(x+2)}\,dx$

PFD

$$(x-6)(x+2)\left[\frac{8}{(x-6)(x+2)} = \frac{A}{x-6} + \frac{B}{x+2}\right](x-6)(x+2)$$

$$8 = A(x+2) + B(x-6)$$
$$= Ax + 2A + Bx - 6B$$
$$= (A+B)x + (2A-6B)$$

Conditions

- $A+B=0 \to B=-A$
- $2A-6B=8$

$$2A-6(-A)=8$$
$$8A=8$$
$$A=1 \Rightarrow B=-1$$

PFD

$$= \lim_{t\to -2^+} \int_t^5 \frac{1}{x-6} - \frac{1}{x+2}\,dx$$

$$= \lim_{t\to -2^+} \ln|x-6| - \ln|x+2| \Big|_t^5$$

$$= \lim_{t\to -2^+} \underbrace{\cancel{\ln 1}}_{0} - \underbrace{\ln 7}_{\text{Finite}} - \underbrace{\cancel{\ln|t-6|}}_{\text{Finite}} + \cancel{\ln|t+2|}$$

$\ln 8$ 8 0^+ $-\infty$

$$= \boxed{-\infty}$$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{4+x^{10}}\,dx = \int_{-\infty}^{0} \frac{x^4}{4+x^{10}}\,dx + \int_0^{\infty} \frac{x^4}{4+x^{10}}\,dx$

$$= \lim_{s\to-\infty} \int_s^0 \frac{x^4}{4+(x^5)^2}\,dx + \lim_{t\to\infty} \int_0^t \frac{x^4}{4+(x^5)^2}\,dx$$

$u = x^5$

$du = 5x^4\,dx$

$\frac{1}{5}\,du = x^4\,dx$

$x=s \Rightarrow u=s^5$

$x=0 \Rightarrow u=0$

$x=t \Rightarrow u=t^5$

$$= \lim_{s\to-\infty} \frac{1}{5}\int_{s^5}^0 \frac{1}{4+u^2}\,du + \lim_{t\to\infty} \frac{1}{5}\int_0^{t^5} \frac{1}{4+u^2}\,du$$

$$= \lim_{s\to-\infty} \frac{1}{5}\left(\frac{1}{2}\right)\arctan\left(\frac{u}{2}\right)\Big|_{s^5}^0 + \lim_{t\to\infty} \frac{1}{5}\left(\frac{1}{2}\right)\arctan\left(\frac{u}{2}\right)\Big|_0^{t^5}$$

$$= \lim_{s\to-\infty} \frac{1}{10}\left[\underbrace{\cancel{\arctan 0}}_{0} - \underbrace{\cancel{\arctan\left(\frac{s^5}{2}\right)}}_{-\pi/2}\right] + \lim_{t\to\infty} \frac{1}{10}\left[\underbrace{\cancel{\arctan\left(\frac{t^5}{2}\right)}}_{\pi/2} - \underbrace{\cancel{\arctan 0}}_{0}\right]$$

$-\infty$ ∞

$$= \frac{1}{10}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \boxed{\frac{\pi}{10}}$$

2. [8 Points] Show that the sequence $\left\{ \left(\frac{n}{n+1}\right)^n \right\}_{n=1}^{\infty}$ converges to $\frac{1}{e}$

1^∞

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = e^{\lim_{x\to\infty} \ln\left[\left(\frac{x}{x+1}\right)^{x}\right]}$$

$\infty \cdot 0$

$$= e^{\lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{1/x}} \quad 0/0$$

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{\left(\frac{x}{x+1}\right)}\left[\frac{\cancel{(x+1)(1)} - \cancel{x(1)}}{(x+1)^2}\right]}{-1/x^2}}$$

$$= e^{\lim_{x\to\infty} \left(\frac{\cancel{x+1}}{\cancel{x}}\right)\left(\frac{1}{(x+1)^{\cancel{2}}}\right)\left(-x^{\cancel{2}}\right)}$$

$$= e^{\lim_{x\to\infty} \frac{-x}{x+1}} \quad -\infty/\infty$$

$$\overset{L'H}{=} e^{\lim_{x\to\infty} \frac{-1}{1}}$$

$$= e^{-1} = \boxed{\frac{1}{e}}$$

3. [15 Points] Consider the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+4n+7}$. Demonstrate **two different** methods to show that this series Diverges.

(a) First, you must use the Integral Test. You can **SKIP** checking the 3 preconditions here. Yay!

(b) Second, use a different method. Your choice.

Complete Square

$$\text{(a)} \int_1^{\infty} \frac{x+1}{x^2+4x+7}\,dx = \lim_{t\to\infty} \int_1^t \frac{x+1}{x^2+4x+7}\,dx = \lim_{t\to\infty} \int_1^t \frac{x+1}{(x+2)^2+3}\,dx$$

$\boxed{\begin{array}{l} u = x+2 \\ du = dx \end{array}}$ $\Rightarrow x = u-2$

$\boxed{\begin{array}{l} x = 1 \Rightarrow u = 3 \\ x = t \Rightarrow u = t+2 \end{array}}$

$$= \lim_{t\to\infty} \int_3^{t+2} \frac{(u-2)+1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \int_3^{t+2} \frac{u-1}{u^2+3}\,du$$

Split-Split

$$= \lim_{t\to\infty} \int_3^{t+2} \frac{u}{u^2+3} - \frac{1}{u^2+3}\,du$$

$$= \lim_{t\to\infty} \frac{1}{2}\ln|u^2+3| - \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t+2}$$

$$= \lim_{t\to\infty} \underbrace{\frac{1}{2}\ln|(t+2)^2+3|}_{\infty} - \underbrace{\frac{1}{\sqrt{3}}\arctan\left(\frac{t+2}{\sqrt{3}}\right)}_{\text{Finite}\ (\pi/2)} - \left[\underbrace{\frac{1}{2}\ln(12)}_{\text{Finite}} - \underbrace{\frac{1}{\sqrt{3}}\arctan\left(\frac{3}{\sqrt{3}}\right)}_{\text{Finite}\ (\pi/3)}\right]$$

$= \boxed{\infty}$ Integral Diverges

$\Rightarrow$ Original Series also Diverges by Integral Test.

3. [15 Points] Consider the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2+4n+7}$. Demonstrate **two different** methods to show that this series Diverges.

(a) First, you must use the Integral Test. You can **SKIP** checking the 3 preconditions here.

(b) Second, use a different method. Your choice.

(b) $$\sum_{n=1}^{\infty} \frac{n+1}{n^2+4n+7} \approx \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$$ Diverges p-Series (Harmonic) $p=1$

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2+4n+7}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2+4n+7} \quad \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{1+\cancel{1/n}^{\,0}}{1+\cancel{4/n}_{\,0}+\cancel{7/n^2}_{\,0}} = 1$$ Finite Non-Zero

$\Rightarrow$ Original Series also Diverges by LCT

4. [26 Points] Determine whether each of the given series **Converges** or **Diverges**. Name any convergence test(s) you use, and justify all of your work.

(a) $\sum_{n=1}^{\infty} n^8+8$ Diverges by nTDT b/c $\lim_{n\to\infty} n^8+8=\infty \neq 0$

(b) $\sum_{n=1}^{\infty} \frac{n^8+8}{n^8+1}$ Diverges by nTDT b/c $\lim_{n\to\infty} \frac{n^8+8}{n^8+1} \frac{1/n^8}{1/n^8} = \lim_{n\to\infty} \frac{1+\cancel{8/n^8}^{\,0}}{1+\cancel{1/n^8}^{\,0}} = 1 \neq 0$

(c) $\sum_{n=1}^{\infty} \frac{1}{n^8+1} \approx \sum_{n=1}^{\infty} \frac{1}{n^8}$ Converges, p-Series $p=8>1$

Bound Terms: $\frac{1}{n^8+1} \leq \frac{1}{n^8} \Rightarrow$ O.S. Converges by CT

(d) $\sum_{n=8}^{\infty} \frac{n^8}{\ln n}$ Diverges by nTDT b/c

$\lim_{n\to\infty} \frac{n^8}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{x^8}{\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{8x^7}{1/x} = \lim_{x\to\infty} 8x^8 = \infty \neq 0$

(e) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^8} \approx \sum_{n=1}^{\infty} \frac{1}{n^8}$ Converges p-Series $p=8>1$

Bound Terms: $\frac{\sin^2 n}{n^8} \leq \frac{1}{n^8} \Rightarrow$ O.S. Converges by CT

(f) $\sum_{n=1}^{\infty} 8$

Diverges by nTDT b/c $\lim_{n\to\infty} 8 = 8 \neq 0$

(g) $\sum_{n=1}^{\infty} \frac{1}{8^n}$ Converges by GST b/c $|r| = |1/8| = \frac{1}{8} < 1$

(h) $\sum_{n=1}^{\infty} \left(\frac{1}{8}+\frac{1}{8^n}\right)$ Diverges by nTDT b/c $\lim_{n\to\infty} \frac{1}{8} + \cancel{\frac{1}{8^n}}^{\,0} = \frac{1}{8} \neq 0$

5. [6 Points] Consider the Series $\sum_{n=1}^{\infty} \frac{8n^2+n}{n^8}$. Show this series Converges by **splitting** it into the sum of two series that are each Convergent. Justify all steps.

$$\sum_{n=1}^{\infty} \frac{8n^2+n}{n^8} = \sum_{n=1}^{\infty} \frac{8n^2}{n^8} + \sum_{n=1}^{\infty} \frac{n}{n^8} = 8\sum_{n=1}^{\infty} \frac{1}{n^6} + \sum_{n=1}^{\infty} \frac{1}{n^7}$$

Constant Multiple of Convergent p-Series $p=6>1$ is Convergent

Convergent p-Series $p=7>1$

O.S. Converges b/c the Sum of 2 Convergent Series Converges

6. [3 Points] Use the Absolute Convergence Test to show that $\sum_{n=1}^{\infty} (-1)^n \frac{8n^2+n}{n^8}$ is convergent. Feel free to reference (not repeat) your work from 8 above.

We showed in 5 that the Absolute Series $\sum \frac{8n^2+n}{n^8}$ Converges, therefore the original Series $\sum (-1)^n \frac{8n^2+n}{n^8}$ Converges by ACT

For 7, 8, and 9, determine whether the given series is **Absolutely Convergent**, **Conditionally Convergent**, or **Divergent**. Name any convergence test(s) you use, and justify all of your work.

7. [3 Points] $\sum_{n=1}^{\infty} (-1)^n \frac{8n^2+n}{n^8}$ Feel free to reference (not repeat) your work above.

Again, We showed in 5 that the Absolute Series $\sum \frac{8n^2+n}{n^8}$ Converges.

$\Rightarrow$ O.S. is Absolutely Convergent by Definition.

A.C.

For 7, 8, and 9, determine whether the given series is **Absolutely Convergent**, **Conditionally Convergent**, or **Divergent**. Name any convergence test(s) you use, and justify all of your work.

8. [7 Points] $\sum_{n=1}^{\infty} \frac{(-1)^n}{8n+3}$ $\xrightarrow{A.S.}$ $\sum_{n=1}^{\infty} \frac{1}{8n+3} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges p-Series $p=1$

AST

① $b_n = \frac{1}{8n+3} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{8n+3} = 0$

③ Terms Decreasing

$b_{n+1} = \frac{1}{8n+11} \leq \frac{1}{8n+3} = b_n$

$\Rightarrow$ Original Series Converges by AST

$\lim_{n\to\infty} \frac{\frac{1}{8n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{8n+3} \cdot \frac{1/n}{1/n}$

$= \lim_{n\to\infty} \frac{1}{8+3/n}$

$= 1/8$ Finite Non-Zero

$\Rightarrow$ Absolute Series also Diverges by LCT

$\Rightarrow$ Original Series Conditionally Convergent by Definition

C.C.

For 7, 8, and 9, determine whether the given series is **Absolutely Convergent**, **Conditionally Convergent**, or **Divergent**. Name any convergence test(s) you use, and justify all of your work.

9. [7 Points] $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (3n)!\ n^8}{8^n\ n^n\ (n!)^2}$ $\leftarrow a_n$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,(3(n+1))!\,(n+1)^8}{8^{n+1}\,(n+1)^{n+1}\,[(n+1)!]^2}}{\dfrac{\cancel{(-1)^n}\,(3n)!\,n^8}{8^n\,n^n\,(n!)^2}}\right|$$

$(3n+3)(3n+2)(3n+1)\cancel{(3n)!}$

$$= \lim_{n\to\infty} \frac{(3n+3)!}{\cancel{(3n)!}} \cdot \cancel{\frac{(n+1)^8}{n^8}} \cdot \frac{\cancel{8^n}}{8^{n+1}} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{(n!)^2}{[(n+1)!]^2}$$

$\left(\frac{n+1}{n}\right)^8 \to 1$ $\quad 8^n \cdot 8$ $\quad (n+1)^n(n+1)$ $\quad (n+1)^2(n!)^2$

$\cancel{\left(1+\frac{1}{n}\right)^8} \to 1$, $\frac{1}{n} \to 0$

$$= \lim_{n\to\infty} \frac{1}{8}\left(\frac{\cancel{3n+3}}{\cancel{n+1}}\right)\left(\frac{3n+2}{n+1}\right)\left(\frac{3n+1}{n+1}\right)\cdot \cancel{\frac{n^n}{(n+1)^n}} \to \frac{1}{e}$$

$3(n+1)$

$$= \lim_{n\to\infty} \frac{3}{8e}\left(\frac{3+2/n}{1+1/n}\right)\left(\frac{3+1/n}{1+1/n}\right) \qquad (\to 3)(\to 3)$$

$$= \frac{27}{8e} > 1$$ Diverges by Ratio Test

$e \approx 2.7$
$e < 3$ $\Big\}$ $8e < 24$
$\Rightarrow 8e < 27$