Exam #2 Review Packet Answer Key

Integrals

1. $$\int_3^\infty \frac{1}{x^2-4x+7}\,dx \underset{\text{Complete Square}}{=} \lim_{t\to\infty}\int_3^t \frac{1}{(x-2)^2+3}\,dx = \lim_{t\to\infty}\int_1^{t-2}\frac{1}{u^2+3}\,du$$

$b^2-4ac = 16-4(1)(7) = \ominus$

$(x-2)^2 = x^2-4x+4$

$\boxed{\begin{array}{l} u=x-2 \\ du=dx \end{array}}$ $\boxed{\begin{array}{l} x=3 \Rightarrow u=1 \\ x=t \Rightarrow u=t-2 \end{array}}$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^{t-2}$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left[\underbrace{\arctan\Big(\underbrace{\tfrac{t-2}{\sqrt{3}}}_{\infty}\Big)}_{\pi/2} - \underbrace{\arctan\left(\tfrac{1}{\sqrt{3}}\right)}_{\pi/6}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right] = \frac{1}{\sqrt{3}}\left[\frac{2\pi}{6}\right] = \boxed{\frac{\pi}{3\sqrt{3}}}$$ Converges

2. $$\int_e^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty}\int_1^{\ln t}\underbrace{\frac{1}{u^3}}_{u^{-3}}\,du = \lim_{t\to\infty}\frac{-1}{2u^2}\Bigg|_1^{\ln t}$$

$\boxed{\begin{array}{l} u=\ln x \\ du=\frac{1}{x}dx \end{array}}$ $\boxed{\begin{array}{l} x=e \Rightarrow u=\ln e=1 \\ x=t \Rightarrow u=\ln t \end{array}}$

$$= \lim_{t\to\infty}\underbrace{\frac{-1}{2\underbrace{(\ln t)^2}_{\infty}}}_{0} + \frac{1}{2} = \boxed{\frac{1}{2}}$$ Converges

3. $$\int_0^\infty \frac{1}{(x-2)(2x+5)}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)(2x+5)}\,dx \underset{\text{PFD}}{=} \lim_{t\to\infty}\int_0^t \frac{1}{x+2}-\frac{2}{2x+5}\,dx$$

P.F.D.

$$\cancel{(x+2)}\cancel{(2x+5)}\left[\frac{1}{\cancel{(x+2)(2x+5)}} = \frac{A}{x+2}+\frac{B}{2x+5}\right](x+2)(2x+5)$$

$$1 = A(2x+5)+B(x+2)$$
$$= 2Ax+5A+Bx+2B$$
$$= (2A+B)x+(5A+2B)$$

$$= \lim_{t\to\infty}\ln|x+2| - \ln|2x+5|\Big|_0^t$$

$$= \lim_{t\to\infty}\ln|t+2|-\ln|2t+5| - \ln 2 + \ln 5$$

$$= \lim_{t\to\infty}\ln\left|\frac{t+2}{2t+5}\,\frac{1/t}{1/t}\right| - \ln 2 + \ln 5$$

Conditions
- $2A+B=0 \Rightarrow B=-2A$
- $5A+2B=1 \quad 5A+2(-2A)=1$

$A=1 \Rightarrow B=-2$

$$= \lim_{t\to\infty} \ln\left|\frac{1+\cancel{2/t}^{\,0}}{2+\cancel{5/t}}\right| - \ln 2 + \ln 5$$

$$= \ln(1/2) - \ln 2 + \ln 5 \quad \text{Converges}$$

$$= \cancel{\ln 1}^{\,0} \underbrace{-\ln 2 - \ln 2}_{-2\ln 2 \to -\ln(2^2) \to -\ln 4} + \ln 5 = \boxed{\ln\left(\frac{5}{4}\right)}$$

4. $$\int_7^\infty \frac{1}{x^2-8x+19}\,dx = \lim_{t\to\infty}\int_7^t \frac{1}{x^2-8x+19}\,dx \overset{\text{Complete Square}}{=} \lim_{t\to\infty}\int_7^t \frac{1}{(x-4)^2+3}\,dx$$

$\boxed{u=x-4 \quad du=dx}$ $\boxed{x=7 \Rightarrow u=3 \quad x=t \Rightarrow u=t-4}$

$$= \lim_{t\to\infty}\int_3^{t-4}\frac{1}{u^2+3}\,du = \lim_{t\to\infty}\frac{1}{\sqrt3}\arctan\left(\frac{u}{\sqrt3}\right)\Bigg|_3^{t-4}$$

$b^2-4ac=64-4(1)(19)=\ominus$

Converges

$$= \lim_{t\to\infty}\frac{1}{\sqrt3}\left[\cancel{\arctan\left(\frac{t-4}{\sqrt3}\right)}^{\pi/2} - \cancel{\arctan\left(\frac{3}{\sqrt3}\right)}^{\pi/3}\right] = \frac{1}{\sqrt3}\left[\frac{\pi}{2}-\frac{\pi}{3}\right] = \frac{1}{\sqrt3}\left[\frac{\pi}{6}\right] = \boxed{\frac{\pi}{6\sqrt3}}$$

$\frac{\pi}{2} \to 3\pi/6$, $\frac{\pi}{3} \to 2\pi/6$

5. $$\int_2^\infty \frac{1}{x^2-2x+4}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x^2-2x+4}\,dx \overset{\text{Complete Square}}{=} \lim_{t\to\infty}\int_2^t \frac{1}{(x-1)^2+3}\,dx$$

x^2-2x+1

$\boxed{u=x-1 \quad du=dx}$ $\boxed{x=2 \Rightarrow u=1 \quad x=t \Rightarrow u=t-1}$

$$= \lim_{t\to\infty}\int_1^{t-1}\frac{1}{u^2+3}\,du = \lim_{t\to\infty}\frac{1}{\sqrt3}\arctan\left(\frac{u}{\sqrt3}\right)\Bigg|_1^{t-1}$$

$b^2-4ac = 4-4(1)(4)=\ominus$

Converges

$$= \lim_{t\to\infty}\frac{1}{\sqrt3}\left[\cancel{\arctan\left(\frac{t-1}{\sqrt3}\right)}^{\pi/2} - \cancel{\arctan\left(\frac{1}{\sqrt3}\right)}^{\pi/6}\right] = \frac{1}{\sqrt3}\left[\frac{2\pi}{6}\right] = \boxed{\frac{\pi}{3\sqrt3}}$$

6. $$\int \frac{1}{x^2+2x+2}\,dx = \int\frac{1}{(x+1)^2+1}\,dx \overset{\text{Complete Square}}{=} \int\frac{1}{u^2+1}\,du = \arctan u + C = \boxed{\arctan(x+1)+C}$$

x^2+2x+1

$b^2-4ac = 4-4(1)(2)=\ominus$

7. $$\int_0^{\pi/2}\tan x\,dx = \lim_{t\to\pi/2^-}\int_0^t \underbrace{\tan x}_{\frac{\sin x}{\cos x}}\,dx = \lim_{t\to\pi/2^-} -\int_1^{\cos t}\frac{1}{u}\,du = \lim_{t\to\pi/2^-} -\ln|u|\Bigg|_1^{\cos t}$$

$u = \cos x$ $\quad x = 0 \Rightarrow u = \cos 0 = 1$

$du = -\sin x\, dx$ $\quad x = t \Rightarrow u = \cos t$

$-du = \sin x\, dx$

$$= \lim_{t \to \frac{\pi}{2}^-} -\ln|\cos t| + \ln 1 = -(-\infty) = \boxed{\infty}$$

Diverges

8. $$\int_3^4 \frac{1}{(x-4)^2}\,dx = \lim_{t \to 4^-} \int_3^t \frac{1}{(x-4)^2}\,dx = \lim_{t \to 4^-} \int_{-1}^{t-4} \frac{1}{u^2}\,du = \lim_{t \to \infty} \left. \frac{-1}{u} \right|_{-1}^{t-4}$$

$u = x-4$, $du = dx$

$x = 3 \Rightarrow u = -1$, $x = t \Rightarrow u = t-4$

$$= \lim_{t \to 4^-} -\frac{1}{t-4} + \frac{1}{-1} = \boxed{\infty}$$

Diverges

9. $$\int_1^2 \frac{1}{x \ln x}\,dx = \lim_{t \to 1^+} \int_t^2 \frac{1}{x \ln x}\,dx = \lim_{t \to 1^+} \int_{\ln t}^{\ln 2} \frac{1}{u}\,du = \lim_{t \to 1^+} \ln|u| \Big|_{\ln t}^{\ln 2}$$

$u = \ln x$, $du = \frac{1}{x}\,dx$

$x = t \Rightarrow u = \ln t$, $x = 2 \Rightarrow u = \ln 2$

$$= \lim_{t \to 1^+} \ln|\ln 2| - \ln|\ln t| = \boxed{\infty}$$

Diverges

10. $$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{t \to 0^+} 2\int_{\sqrt{t}}^1 e^u\,du = \lim_{t \to 0^+} 2e^u \Big|_{\sqrt{t}}^1$$

$u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$, $2du = \frac{1}{\sqrt{x}}\,dx$

$x = t \Rightarrow u = \sqrt{t}$, $x = 1 \Rightarrow \sqrt{1} = 1$

$$= \lim_{t \to 0^+} 2\left[e - e^{\sqrt{t}}\right] = \boxed{2e - 2}$$

Converges

11. $$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t \to \infty} \int_0^{\ln t} u\,du = \lim_{t \to \infty} \left. \frac{u^2}{2} \right|_0^{\ln t}$$

$u = \ln x$, $du = \frac{1}{x}\,dx$

$x = 1 \Rightarrow u = \ln 1 = 0$, $x = t \Rightarrow u = \ln t$

$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2} - 0 = \boxed{\infty}$$

Diverges

12. $\displaystyle\int \frac{2x-5}{x^2+2x+2}\,dx = \int \frac{2x-5}{(x+1)^2+1}\,dx = \int \frac{\overbrace{2(u-1)-5}^{2u-2-5}}{u^2+1}\,du = \int \frac{2u-7}{u^2+1}\,du$ Split·Split

$b^2-4ac = 4-4(1)(2) = \ominus$

$(x+1)^2 = x^2+2x+1$

$u = x+1 \Rightarrow x = u-1$
$du = dx$

$= 2\displaystyle\int \frac{u}{u^2+1} - 7\int \frac{1}{u^2+1}\,du$

$= \ln|u^2+1| - 7\arctan u + C$

$= \boxed{\ln|\underbrace{(x+1)^2+1}_{x^2+2x+2}| - 7\arctan(x+1) + C}$

13. $\displaystyle\int_0^1 \frac{e^x}{\sqrt{e^x-1}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{e^x}{\sqrt{e^x-1}}\,dx = \lim_{t\to 0^+} \int_{e^t-1}^{e-1} \frac{1}{\sqrt{u}}\,du = \lim_{t\to 0^+} 2\sqrt{u}\Big|_{e^t-1}^{e-1}$ $u^{-1/2}$

$u = e^x-1$
$du = e^x\,dx$

$x = t \Rightarrow u = e^t-1$
$x = 1 \Rightarrow u = e-1$

$= \displaystyle\lim_{t\to 0^+} 2\sqrt{e-1} - 2\sqrt{e^t-1} = \boxed{2\sqrt{e-1}}$ Converges

14. $\displaystyle\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x \cdot 1\,dx = \lim_{t\to 0^+} x\ln x\Big|_t^1 - \int_t^1 1\,dx$

Converges

$u = \ln x$ $dv = 1\,dx$
$du = \frac{1}{x}\,dx$ $v = x$

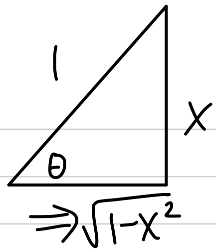
$= \displaystyle\lim_{t\to 0^+} x\ln x\Big|_t^1 - x\Big|_t^1 = \lim_{t\to 0^+} \cancel{\ln 1} - \cancel{t\ln t} - (1-\cancel{t}) = \boxed{-1}$ $0 \cdot (-\infty)$ $(*)$

$(*)$ $\displaystyle\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} \overset{LH}{=} \lim_{t\to 0^+} \frac{\frac{1}{t}(-t^2)}{-\frac{1}{t^2}} = \lim_{t\to 0^+} -t = 0$ $0\cdot(-\infty)$ $\frac{-\infty}{\infty}$

15. $\displaystyle\int_0^1 \frac{1}{(1-x^2)^{3/2}}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{(1-x^2)^{3/2}} = \lim_{t\to 1^-} \int_0^t \frac{1}{(\sqrt{1-x^2})^3}\,dx$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$

$= \displaystyle\lim_{t\to 1^-} \int_{x=0}^{x=t} \frac{1}{(\sqrt{1-\sin^2\theta})^3} \cdot \cos\theta\,d\theta = \lim_{t\to 1^-} \int_{x=0}^{x=t} \frac{1}{\cos^2\theta}\,d\theta$



$\left(\sqrt{\cos^2\theta}\right)^3 \to \cos^3\theta$

$$= \lim_{t\to 1^-} \int_{x=0}^{x=t} \sec^2\theta \, d\theta = \lim_{t\to 1^-} \tan\theta \Big|_{x=0}^{x=t}$$

$$= \lim_{t\to 1^-} \frac{x}{\sqrt{1-x^2}} \Bigg|_0^t = \lim_{t\to 1^-} \frac{\overset{1}{\cancel{t}}}{\underbrace{\sqrt{\cancel{1-t^2}}}_{\to 0^+}} - 0 = \boxed{\infty} \quad \text{Diverges}$$

16. $$\int_{-\infty}^{\infty} \frac{1}{x^2-6x+10}\,dx = \int_{-\infty}^{0} \frac{1}{x^2-6x+10}\,dx + \int_0^{\infty} \frac{1}{x^2-6x+10}\,dx$$

$b^2-4ac = 36-4(1)(10)$

$$= \lim_{s\to-\infty} \int_s^0 \frac{1}{x^2-6x+10}\,dx + \lim_{t\to\infty} \int_0^t \frac{1}{x^2-6x+10}\,dx$$

Complete Sq.

$$= \lim_{s\to-\infty} \int_s^0 \frac{1}{(x-3)^2+1}\,dx + \lim_{t\to\infty} \int_0^t \frac{1}{(x-3)^2+1}\,dx$$

$\boxed{\begin{array}{l} u = x-3 \\ du = dx \end{array}}$

$\boxed{\begin{array}{l} x = s \Rightarrow u = s-3 \\ x = 0 \Rightarrow u = -3 \\ x = t \Rightarrow u = t-3 \end{array}}$

$$= \lim_{s\to-\infty} \int_{s-3}^{-3} \frac{1}{u^2+1} + \lim_{t\to\infty} \int_{-3}^{t-3} \frac{1}{u^2+1}\,du$$

OR split at 3 could have made nice values

$$= \lim_{s\to-\infty} \arctan u \Big|_{s-3}^{-3} + \lim_{t\to\infty} \arctan u \Big|_{-3}^{t-3}$$

$$= \lim_{s\to-\infty} \cancel{\arctan(-3)} - \underbrace{\arctan(\overset{-\infty}{s-3})}_{-\pi/2} + \lim_{t\to\infty} \underbrace{\arctan(\overset{\infty}{t-3})}_{\pi/2} - \cancel{\arctan(-3)}$$

cancel

$$= \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

Converges

Both Pieces Finite ADD them

17. $\displaystyle\int_{-5}^{0} \frac{x}{x^2+4x-5}\,dx = \int_{-5}^{0} \frac{x}{\underbrace{(x+5)}_{x=-5}(x-1)}\,dx = \lim_{t\to -5^+} \int_t^0 \frac{x}{(x+5)(x-1)}\,dx$

PFD

$$(x+5)(x-1)\left[\frac{x}{\cancel{(x+5)(x-1)}} = \frac{A}{x+5} + \frac{B}{x-1}\right](x+5)(x-1)$$

$x = A(x-1) + B(x+5)$

$= Ax - A + Bx + 5B$

$= (A+B)x + 5B - A$

Conditions

- $A+B=1 \Rightarrow B = 1-A$
- $5B - A = 0$

$5(1-A) - A = 0$

$5 - 5A - A = 0$

$6A = 5$

$A = 5/6 \Rightarrow B = 1 - 5/6 = 1/6$

$= \displaystyle\lim_{t\to -5^+} \int_t^0 \frac{\frac{5}{6}}{(x+5)} + \frac{\frac{1}{6}}{x-1}\,dx$

$= \displaystyle\lim_{t\to -5^+} \frac{5}{6}\ln|x+5| + \frac{1}{6}\ln|x-1| \Big|_t^0$

$= \displaystyle\lim_{t\to -5^+} \frac{5}{6}\ln 5 - \frac{1}{6}\cancel{\ln 1}^{\,0} - \left(\frac{5}{6}\cancel{\ln|t+5|}^{\,-\infty} + \frac{1}{6}\cancel{\ln|t-1|}^{\,\ln 6}\right)$

0^+

$= \boxed{\infty}$ Diverges

18. $\displaystyle\int \frac{x^5+2}{x^2-1}\,dx = \int x^3 + x + \frac{x+2}{\underbrace{x^2-1}_{(x-1)(x+1)}}\,dx = \int x^3 + x + \frac{3/2}{x-1} - \frac{\frac{1}{2}}{x+1}\,dx$

Long Division

$$\begin{array}{r} x^3 + x \\ x^2-1 \,\big)\, \overline{x^5 + 2} \\ -(\cancel{x^5} - x^3) \\ \hline x^3 + 2 \\ -(\cancel{x^3} - x) \\ \hline x + 2 \end{array}$$

$= \boxed{\dfrac{x^4}{4} + \dfrac{x^2}{2} + \dfrac{3}{2}\ln|x-1| - \dfrac{1}{2}\ln|x+1| + C}$

PFD

$$\cancel{(x-1)(x+1)}\left[\frac{x+2}{\cancel{(x-1)(x+1)}} = \frac{A}{x-1} + \frac{B}{x+1}\right](x-1)(x+1)$$

$x + 2 = A(x+1) + B(x-1)$

$= Ax + A + Bx - B$

$= (A+B)x + A - B$

Conditions

- $A+B=1 \Rightarrow A = 1-B$
- $A-B=2$

$1-B-B=2$

$1-2B=2$

$2B=-1$

$B = -1/2 \Rightarrow A = 1-(-1/2) = 3/2$

19. $\displaystyle\int_0^6 \frac{1}{(x-2)^2}\,dx = \int_0^{2} \frac{1}{(x-2)^2}\,dx + \int_{2}^6 \frac{1}{(x-2)^2}\,dx$

$\displaystyle = \lim_{s\to 2^-} \int_0^s \frac{1}{(x-2)^2}\,dx + \lim_{t\to 2^+} \int_t^6 \frac{1}{(x-2)^2}\,dx$

↳ u-sub?!

$\displaystyle = \lim_{s\to 2^-} -\frac{1}{x-2}\Big|_0^s + \lim_{t\to 2^+} -\frac{1}{x-2}\Big|_t^6$

$\displaystyle = \lim_{s\to 2^-} \cancel{-\frac{1}{s-2}} + \frac{1}{-2} + \lim_{t\to 2^+} -\frac{1}{4} + \cancel{\frac{1}{t-2}}$

∞, $s \to 2^-$, $s-2 \to 0^-$; $+\infty$, $t \to 2^+$, $t-2 \to 0^+$

$+\infty$ — Diverges

$+\infty$ here — Diverges

Original Integral $\boxed{\text{Diverges}}$ b/c one piece Diverges (BOTH do...)

Note: Both pieces finished to check answers... but you only need to finish one piece if Divergent.

20. $\displaystyle\int_0^{\infty} \frac{1}{x^2+3x+2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+3x+2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{(x+2)(x+1)}\,dx$

$\displaystyle = \lim_{t\to\infty} \int_0^t \frac{-1}{x+2} + \frac{1}{x+1}\,dx$

PFD

$\displaystyle (x+2)(x+1)\left[\frac{1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}\right](x+2)(x+1)$

$1 = A(x+1) + B(x+2)$

$= Ax + A + Bx + 2B$

$= (A+B)x + A + 2B$

Conditions

$\displaystyle = \lim_{t\to\infty} -\ln|x+2| + \ln|x+1| \Big|_0^t$

$\displaystyle = \lim_{t\to\infty} \underbrace{-\ln|t+2| + \ln|t+1|}_{\text{Indeterminate Difference}} - \left(-\ln 2 + \cancel{\ln 1}\right)$

$-\infty$, ∞, 0

• $A+B=0 \Rightarrow A=-B$

• $A+2B=1$ $\quad -B+2B=1$

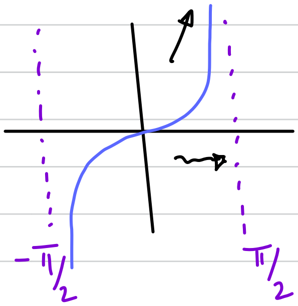
$B=1 \Rightarrow A=-1$

$$= \lim_{t\to\infty} \ln\left|\frac{t+1}{t+2}\,\frac{1/t}{1/t}\right| + \ln 2$$

$$= \lim_{t\to\infty} \underbrace{\cancel{\ln\left|\frac{1+\cancel{1/t}^{\,0}}{1+\cancel{2/t}^{\,0}}\right|}}_{\ln 1 \to 0} + \ln 2 = \boxed{\ln 2}$$

Converges

21. $$\int_0^{\pi/2} \tan^2 x\,dx = \lim_{t\to\pi/2^-}\int_0^t \tan^2 x\,dx = \lim_{t\to\pi/2^-}\int_0^t \sec^2 x - 1\,dx$$



$$= \lim_{t\to\pi/2^-} \tan x - x\,\Big|_0^t = \lim_{t\to\pi/2^-} \cancel{\tan t}^{\,+\infty} - \cancel{t}^{\,\pi/2} - (\cancel{\tan 0 - 0})^{\,0}$$

$$= \boxed{\infty} \quad \text{Diverges}$$

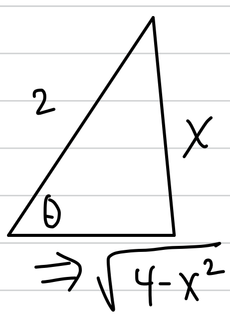
22. $$\int_0^{2} \frac{1}{(4-x^2)^{3/2}}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{\left(\sqrt{4-x^2}\right)^3}\,dx = \lim_{t\to 2^-}\int_{x=0}^{x=t} \frac{1}{\left(\sqrt{\underbrace{4-4\sin^2\theta}}\right)^3}\,2\cos\theta\,d\theta$$

$\sqrt{4(1-\sin^2\theta)}$

$\sqrt{4\cos^2\theta} \to (2\cos\theta)^3$

$$\boxed{\begin{aligned} x&=\sin\theta \\ dx&=\cos\theta\,d\theta \end{aligned}}$$

$$= \lim_{t\to 2^-} \frac{1}{4}\int_{x=0}^{x=t} \frac{1}{\cos^2\theta}\,d\theta = \lim_{t\to 2^-} \frac{1}{4}\int_{x=0}^{x=t} \sec^2\theta\,d\theta$$



$$= \lim_{t\to 2^-} \frac{1}{4}\tan\theta\,\Big|_{x=0}^{x=t} = \lim_{t\to 2^-} \frac{1}{4}\left(\frac{x}{\sqrt{4-x^2}}\right)\Big|_{x=0}^{x=t}$$

$$= \lim_{t\to 2^-} \frac{1}{4}\left(\cancel{\frac{t^{\,\to 2}}{\sqrt{4-t^2}_{\,\to 0^+}}}^{\,\to +\infty}\right) - \cancel{\frac{1}{4}\cdot 0}^{\,0} = \boxed{\infty} \quad \text{Diverges}$$

23. $\displaystyle\int \frac{4x^2+7x+6}{(x+2)(x^2+4)}\,dx = \int \frac{1}{x+2} + \frac{3x+1}{x^2+4}\,dx = \int \frac{1}{x+2} + \frac{3x}{x^2+4} + \frac{1}{x^2+4}\,dx$

$$\boxed{= \ln|x+2| + \frac{3}{2}\ln|x^2+4| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

PFD

$$(x+2)(x^2+4)\left[\frac{4x^2+7x+6}{\cancel{(x+2)(x^2+4)}} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}\right](x+2)(x^2+4)$$

$$4x^2+7x+6 = A(x^2+4)+(Bx+C)(x+2)$$
$$= Ax^2+4A+Bx^2+2Bx+Cx+2C$$
$$= (A+B)x^2+(2B+C)x+4A+2C$$

Conditions

- $A+B=4 \Rightarrow B=4-A$
- $2B+C=7$ $\quad 2(4-A)+C=7 \Rightarrow 8-2A+C=7$
- $4A+2C=6$ $\quad C=2A-1$

$$4A+2(2A-1)=6$$
$$4A+4A-2=6$$
$$8A=8$$
$$C=2-1=1 \Leftarrow A=1 \Rightarrow B=4-1=3$$

24. $\displaystyle\int_1^{\infty} \frac{1}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x} - \frac{1}{x+1}\,dx$

PFD

$$x(x+1)\left[\frac{1}{\cancel{x(x+1)}} = \frac{A}{x} + \frac{B}{x+1}\right]x(x+1)$$

$$1 = A(x+1)+Bx$$
$$= (A+B)x+A$$

Conditions

- $A+B=0$
- $A=1 \Rightarrow B=-1$

$$= \lim_{t\to\infty} \ln|x| - \ln|x+1| \Big|_1^t$$

$$= \lim_{t\to\infty} \underbrace{\ln|t| - \ln|t+1|}_{\text{Indeterminate Difference}} - (\ln 1 - \ln 2)$$

∞ $\quad\infty\quad$ 0

$$= \lim_{t\to\infty} \ln\left|\frac{t \cdot \frac{1}{t}}{(t+1)\cdot\frac{1}{t}}\right| + \ln 2$$

OR L'H

0

$$= \lim_{t\to\infty} \ln\left|\frac{1}{1+\frac{1}{t}}\right| + \ln 2 = \boxed{\ln 2} \quad \text{Converges}$$

25. $$\int_{-3}^{-2} \frac{1}{x^2-4}\,dx = \int_{-3}^{-2} \frac{1}{(x-2)(x+2)}\,dx = \lim_{t\to -2^-} \int_{-3}^{t} \frac{1}{(x-2)(x+2)}\,dx$$

$x=-2$

PFD

$$(x-2)(x+2)\left[\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}\right](x-2)(x+2)$$

$$1 = A(x+2) + B(x-2)$$
$$= Ax + 2A + Bx - 2B$$
$$= (A+B)x + 2A - 2B$$

Conditions
- $A+B=0 \Rightarrow B=-A$
- $2A-2B=1$

$$2A-2(-A)=1$$
$$4A=1$$
$$A=\tfrac{1}{4} \Rightarrow B=-\tfrac{1}{4}$$

$$= \lim_{t\to -2^-} \int_{3}^{t} \frac{1/4}{x-2} - \frac{1/4}{x+2}\,dx$$

$$= \lim_{t\to -2^-} \tfrac{1}{4}\ln|x-2| - \tfrac{1}{4}\ln|x+2| \Big|_3^t$$

$$= \lim_{t\to -2^-} \frac{1}{4}\ln|t-2| - \frac{1}{4}\ln|t+2| - \left(\frac{1}{4}\ln|1 - \frac{1}{4}\ln 5\right)$$

$\ln 4$, $|-4|$, $|0^-|$, 0^+, $-\infty$, 0

$$= \boxed{\infty} \quad \text{Diverges}$$

26. $$\int_0^1 \arcsin x\,dx = \int_0^1 \arcsin x \cdot 1\,dx = x\arcsin x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$$

IBP

$u = \arcsin x$	$dv = 1\,dx$
$du = \frac{1}{\sqrt{1-x^2}}\,dx$	$v = x$

$u = 1-x^2$
$du = -2x\,dx$
$-\frac{1}{2}du = x\,dx$

$x=0 \Rightarrow u=1$
$x=t \Rightarrow u=1-t^2$

$$= x\arcsin x\Big|_0^1 - \lim_{t\to 1^-} \int_0^t \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\arcsin x\Big|_0^1 + \lim_{t\to 1^-} \frac{1}{2}\int_1^{1-t^2} \frac{1}{\sqrt{u}}\,du$$

$$= x\arcsin x\Big|_0^1 + \lim_{t\to 1^-} \sqrt{u}\Big|_1^{1-t^2}$$

$$= \arcsin 1 - 0 + \lim_{t\to 1^-} \sqrt{1-t^2} - \sqrt{1} = \boxed{\tfrac{\pi}{2} - 1} \quad \text{Converges}$$

$\arcsin 1 \to \pi/2$, $\sqrt{1-t^2}\to 0$ (0^+), $\sqrt{1}\to 1$

27. $\int \frac{x^3+7x+1}{x^2+1}\,dx = \int x + \frac{6x+1}{x^2+1}\,dx = \int x + \frac{6x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$$= \boxed{\frac{x^2}{2} + 3\ln|x^2+1| + \arctan x + C}$$

Long Division

$$\begin{array}{r} x \\ x^2+1 \overline{) x^3+7x+1} \\ -(x^3 + x) \\ \hline 6x+1 \end{array}$$

28. $\int \frac{x^4+x^3+2x^2+6x+2}{\underbrace{(x+1)(x^2+1)}_{x^3+x^2+x+1}}\,dx = \int x + \frac{x^2+5x+2}{(x+1)(x^2+1)}\,dx$

$$= \int x - \frac{1}{x+1} + \frac{2x+3}{x^2+1}\,dx$$

$$= \boxed{\frac{x^2}{2} - \ln|x+1| + \ln|x^2+1| + 3\arctan x + C}$$

Long Division

$$\begin{array}{r} x \\ x^3+x^2+x+1 \overline{) \cancel{x^4}+\cancel{x^3}+2x^2+6x+2} \\ -(\cancel{x^4}+\cancel{x^3}+x^2+x) \\ \hline x^2+5x+2 \end{array}$$

PFD

$$\cancel{(x+1)(x^2+1)}\left[\frac{x^2+5x+2}{\cancel{(x+1)(x^2+1)}} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right](x+1)(x^2+1)$$

$$x^2+5x+2 = A(x^2+1) + (Bx+C)(x+1)$$
$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$
$$= (A+B)x^2 + (B+C)x + A+C$$

Conditions

- $A+B=1 \Rightarrow B=1-A$
- $B+C=5 \quad 1-A+C=5$
- $A+C=2 \quad \hookrightarrow C=4+A$

$$A+4+A=2$$
$$2A=-2$$
$$A=-1 \Rightarrow B=1-(-1)=2$$
$$\Rightarrow C=4-1=3$$

29. $$\int_0^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^+} -\int_{1/t}^1 e^u\,du$$

$u = 1/x$

$du = -1/x^2\,dx$

$-du = \frac{1}{x^2}\,dx$

$x = t \Rightarrow u = 1/t$

$x = 1 \Rightarrow u = 1$

$$= \lim_{t\to 0^+} -e^u\Big|_{1/t}^{1} = \lim_{t\to 0^+} -e + \overbrace{e^{\overbrace{1/t}^{\infty}}}^{+\infty}$$

$$= \boxed{\infty} \qquad \text{Diverges}$$

Same u-sub

30. $$\int_{-1}^0 \frac{e^{1/x}}{x^2}\,dx = \lim_{t\to 0^-}\int_{-1}^t \frac{e^{1/x}}{x^2}\,dx = \dots = \lim_{t\to 0^-} -e^{1/x}\Big|_{-1}^{t}$$

$$= \lim_{t\to 0^-} -\underbrace{e^{\overbrace{1/t}^{-\infty}}}_{0} + e^{-1} = \boxed{\frac{1}{e}} \qquad \text{Converges}$$

31. $$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^{\infty} \frac{x^2}{9+x^6}\,dx$$

$$= \lim_{s\to-\infty}\int_s^0 \frac{x^2}{9+\underbrace{x^6}_{(x^3)^2}}\,dx + \lim_{t\to\infty}\int_0^t \frac{x^2}{9+\underbrace{x^6}_{(x^3)^2}}\,dx$$

$u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3}\,du = x^2\,dx$

$x = s \Rightarrow u = s^3$

$x = 0 \Rightarrow u = 0$

$x = t \Rightarrow u = t^3$

$$= \lim_{s\to-\infty} \frac{1}{3}\int_{s^3}^0 \frac{1}{9+u^2}\,du + \frac{1}{3}\int_0^{t^3} \frac{1}{9+u^2}\,du$$

$$= \lim_{s\to-\infty} \frac{1}{3}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)\Big|_{s^3}^{0} + \lim_{t\to\infty}\frac{1}{3}\cdot\frac{1}{3}\arctan\left(\frac{u}{3}\right)\Big|_0^{t^3}$$

$$= \lim_{s\to-\infty}\frac{1}{9}\Big[\overbrace{\arctan 0}^{0} - \overbrace{\arctan\Big(\overbrace{\tfrac{s^3}{3}}^{-\infty}\Big)}^{-\pi/2}\Big] + \lim_{t\to\infty}\frac{1}{9}\Big[\overbrace{\arctan\Big(\overbrace{\tfrac{t^3}{3}}^{\infty}\Big)}^{\pi/2} - \overbrace{\arctan 0}^{0}\Big]$$

$$= \frac{1}{9}\left(\frac{\pi}{2}\right) + \frac{1}{9}\left(\frac{\pi}{2}\right) = \boxed{\frac{\pi}{9}}$$ Converges

32. $$\int_2^{\infty} \frac{x}{e^{3x}}\,dx = \lim_{t\to\infty} \int_2^t x e^{-3x}\,dx = \lim_{t\to\infty} \left.\frac{-x}{3e^{3x}}\right|_2^t + \frac{1}{3}\int_2^t e^{-3x}\,dx \rightarrow \frac{e^{-3x}}{-3}$$

IBP

$$\boxed{\begin{array}{ll} u = x & dv = e^{-3x}\,dx \\ du = dx & v = \frac{e^{-3x}}{-3} \end{array}}$$

$$= \lim_{t\to\infty} \left.\frac{-x}{3e^{3x}}\right|_2^t - \left.\frac{1}{9e^{3x}}\right|_2^t$$

$$= \lim_{t\to\infty} \overset{-\infty/\infty}{\frac{-t}{3e^{3t}}} + \frac{2}{3e^6} - \overset{0}{\cancel{\frac{1}{9e^{3t}}}} + \frac{1}{9e^6}$$

$$\overset{}{\underset{\text{L'H}}{=}} \lim_{t\to\infty} \overset{0}{\cancel{\frac{-1}{9e^{3t}}}} + \frac{6}{9e^6} + \frac{1}{9e^6} = \boxed{\frac{7}{9e^6}}$$ Converges

33. $$\int_0^e \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^e \ln x \cdot x^{-1/2}\,dx = \lim_{t\to 0^+} \left. 2\sqrt{x}\ln x\right|_t^e - 2\int_t^e \frac{1}{\sqrt{x}}\,dx$$ $x^{-1/2}$

IBP

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^{-1/2}\,dx \\ du = \frac{1}{x}\,dx & v = 2\sqrt{x} \end{array}}$$

$$= \lim_{t\to 0^+} \left. 2\sqrt{x}\ln x\right|_t^e - \left. 4\sqrt{x}\right|_t^e$$

$$= \lim_{t\to 0^+} 2\sqrt{e}\,\overset{1}{\cancel{\ln e}} - \underbrace{\overset{0\cdot(-\infty)}{\cancel{2\sqrt{t}\ln t}}}_{(*)\ \to 0} - 4\sqrt{e} + \overset{0}{\cancel{4\sqrt{t}}}$$

$$= 2\sqrt{e} - 4\sqrt{e} = \boxed{-2\sqrt{e}}$$ Converges

(*) $$\lim_{t\to 0^+} \overset{0\cdot(-\infty)}{\sqrt{t}\cdot\ln t} = \underset{\text{L'H}}{\lim_{t\to 0^+}} \overset{-\infty/\infty}{\frac{\ln t}{1/\sqrt{t}}} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{\frac{-1}{2t^{3/2}}} \overset{-2t^{3/2}}{} = \lim_{t\to 0^+} -\overset{0}{\cancel{2\sqrt{t}}} = 0$$

$t^{-1/2} \rightarrow -\frac{1}{2}t^{-3/2}$

Sequences

simplify $\frac{n}{n^4}$ or indeterminate ∞/∞

34. $$\lim_{n\to\infty} \frac{1+n-7n^4}{3n^4+8n^3+9}\ \frac{1/n^4}{1/n^4} = \lim_{n\to\infty} \frac{\cancel{\frac{1}{n^4}}^{\to 0}+\cancel{\frac{1}{n^3}}^{\to 0}-7}{3+\cancel{\frac{8}{n}}_{\to 0}+\cancel{\frac{9}{n^4}}_{\to 0}} = \boxed{\frac{-7}{3}}$$ Converges

35. $$\lim_{n\to\infty} \frac{n^3}{(n+1)^3} \overset{\infty/\infty}{=} \lim_{n\to\infty}\left(\frac{n}{n+1}\ \frac{1/n}{1/n}\right)^3 = \lim_{n\to\infty}\cancel{\left(\frac{1}{1+\cancel{1/n}_{\to 0}}\right)^3}^{\to 1} = \boxed{1}$$ Converges

36. $$\lim_{n\to\infty}\left(\frac{n-5}{n}\right)^n \overset{1^\infty}{=} \lim_{n\to\infty}\left(1-\frac{5}{n}\right)^n = \lim_{x\to\infty}\left(1-\frac{5}{x}\right)^x = e^{\lim_{x\to\infty}\ln\left[\left(1-5/x\right)^x\right]}$$

$$= e^{\lim_{x\to\infty} x\ln(1-5/x)}\ (\infty\cdot 0) = e^{\lim_{x\to\infty}\frac{\ln(1-5/x)}{1/x}}\ (0/0)$$ Converges

$$\overset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{1-5/x}\left(\cancel{\frac{5}{x^2}}\right)}{\cancel{-1/x^2}}} = e^{\lim_{x\to\infty}\cancel{\frac{1}{1-\cancel{5/x}_{\to 0}}}^{\to 1}(-5)} = \boxed{e^{-5}}$$

37. $$\lim_{n\to\infty}\frac{(2n+3)!}{(2n+5)!} = \lim_{n\to\infty}\frac{\cancel{(2n+3)!}}{(2n+5)(2n+4)\cancel{(2n+3)!}} = \lim_{n\to\infty}\cancel{\frac{1}{(2n+5)(2n+4)}}^{\to 0}$$

$$= \boxed{0}$$ Converges

OR expand
$$= \lim_{n\to\infty}\frac{\cancel{(2n+3)(2n+2)(2n+1)(2n)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}}{(2n+5)(2n+4)\cancel{(2n+3)(2n+2)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1}} = \boxed{0}$$ same

38. $$\lim_{n\to\infty}\cancel{\arctan(n^2+1)}_{\to \pi/2} = \boxed{\frac{\pi}{2}}$$ Converges

39. $\lim_{n\to\infty} \frac{\sqrt{n}}{(\ln n)^2} \;\; \infty/\infty \;\; = \lim_{x\to\infty} \frac{\sqrt{x}}{(\ln x)^2} \;\; \infty/\infty \;\; \underset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{2(\ln x)\cdot\frac{1}{x}} \quad \frac{\cancel{x}\,\sqrt{x}}{\cancel{2\sqrt{x}}\;2\ln x}$

$= \lim_{x\to\infty} \frac{\sqrt{x}}{4\ln x} \;\; \infty/\infty \;\; \underset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{4}{x}} \;\; x/4 \;\; = \lim_{x\to\infty} \frac{\sqrt{x}\to\infty}{8} = \boxed{\infty}$

Diverges

40. $\lim_{n\to\infty} n^{1/n} \;\; \infty^0 \;\; = \lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \ln\left[x^{1/x}\right]} = e^{\lim_{x\to\infty} \frac{1}{x}\cdot\ln x}$

$= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \;\; \infty/\infty \;\; \underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{x}}{1}} = e^0 = \boxed{1}$ Converges

41. $\lim_{n\to\infty} n\cdot\sin(1/n) \;\; \infty\cdot 0 \;\; = \lim_{x\to\infty} x\cdot\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \;\; 0/0$

$= \lim_{x\to\infty} \frac{\cos(1/x)\,\cancel{(-1/x^2)}}{\cancel{-1/x^2}} = \lim_{x\to\infty} \cos(1/x) \to 1 = \boxed{1}$ Converges

42. $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \;\; 1^\infty \;\; = \lim_{x\to\infty} \left(\frac{x+1}{x}\right)^x = \lim_{x\to\infty} \left(1+1/x\right)^x = e^{\lim_{x\to\infty} \ln\left[\left(1+1/x\right)^x\right]}$

$= e^{\lim_{x\to\infty} x\ln(1+1/x)} = e^{\lim_{x\to\infty} \frac{\ln(1+1/x)}{1/x}} \;\; 0/0$

$\underset{L'H}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cancel{(-1/x^2)}}{\cancel{-1/x^2}}} = \boxed{e}$ ← Learn for Ratio Test.

$$43.\ \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = e^{\lim_{x\to\infty}\ln\left[\left(\frac{x}{x+1}\right)^x\right]}$$

1^∞

$\infty\cdot 0$

$$= e^{\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)} = e^{\lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{1/x}}$$

$0/0$

$$\underset{L'H}{=} e^{\lim_{x\to\infty}\frac{\frac{1}{\left(\frac{x}{x+1}\right)}\left[\frac{(x+1)(1)-x(1)}{(x+1)^2}\right]}{-1/x^2}}$$

$-x^2$

$$= e^{\lim_{x\to\infty}\frac{x+1}{\cancel{x}}\left[\frac{1}{(x+1)^{\cancel{2}}}\right]\cdot(-x^{\cancel{2}})}$$

$-\infty/\infty$

$$= e^{\lim_{x\to\infty}\frac{-x}{x+1}} \underset{L'H}{=} e^{\lim_{x\to\infty}\frac{-1}{1}} = e^{-1} = \boxed{\frac{1}{e}}$$

Converges

or $\frac{1/x}{1/x}$

Series Sums

$$44.\ \sum_{n=1}^{\infty}\frac{2^n+3^n}{6^n} = \sum_{n=1}^{\infty}\frac{2^n}{6^n} + \sum_{n=1}^{\infty}\frac{3^n}{6^n} = \sum_{n=1}^{\infty}\left(\frac{2}{6}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{6}\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{1}{3^n} + \sum_{n=1}^{\infty}\frac{1}{2^n}$$

$$= \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots \qquad = \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$$

$a = 1/3$

$r = 1/3$

Conv. GST $|r| = 1/3 < 1$

Conv. GST $|r| = 1/2 < 1$

$a = 1/2$

$r = 1/2$

$$\text{Sum} = \frac{a}{1-r} = \frac{1/3}{1-1/3} = \frac{1/\cancel{3}}{2/\cancel{3}} = \frac{1}{2} + \text{Sum} = \frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{\cancel{1/2}}{\cancel{1/2}} = 1$$

$\frac{1}{2}+1=\boxed{\frac{3}{2}}$

Sum of 2 Convergent Series **is** Convergent

not needed here b/c didn't ask to **prove** Converge

45. $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\,2^{n-1}}{3^{n+1}} = \frac{2^0}{3^2}-\frac{2^1}{3^3}+\frac{2^2}{3^4}+\dots$$

$n=1$ $n=2$ $n=3$

$\frac{2^0}{3^2} \rightarrow a$ $r=\frac{2}{3}$

$a=\frac{1}{9}$

$r=-\frac{2}{3}$

[Convergent GST $|r|=\left|-\frac{2}{3}\right|=\frac{2}{3}<1$] not needed for Just Sum

$$\text{Sum}=\frac{a}{1-r}=\frac{\frac{1}{9}}{1-\left(-\frac{2}{3}\right)}=\frac{\frac{1}{\cancel{9}\,3}}{\frac{5}{3}} \cdot \frac{\cancel{3}}{5}=\boxed{\frac{1}{15}}$$

46. $$\sum_{n=1}^{\infty}\frac{(-1)^n\,3^{n+2}}{2^{4n-1}}=\frac{-3^3}{2^3}+\frac{3^4}{2^7}-\frac{3^5}{2^{11}}+\dots$$

$n=1$ $n=2$ $n=3$

$\frac{-3^3}{2^3} \rightarrow a$ $r=\frac{-3}{2^4}$

not needed for Just Sum

$a=\frac{-3^3}{2^3}=\frac{-27}{8}$

$r=-\frac{3}{2^4}=\frac{-3}{16}$

Convergent GST $|r|=\left|-\frac{3}{16}\right|=\frac{3}{16}<1$

$$\text{Sum}=\frac{a}{1-r}=\frac{\frac{-27}{8}}{1-\left(-\frac{3}{16}\right)}=\frac{\frac{-27}{\cancel{8}}}{\frac{19}{16}} \cdot \frac{\cancel{16}\,2}{19}=\boxed{\frac{-54}{19}}$$

47. $$\sum_{n=1}^{\infty}\frac{(-1)^n\,4^n}{3^{2n-1}}=\frac{-4}{3^1}+\frac{4^2}{3^3}-\frac{4^3}{3^5}+\dots$$

$n=1$ $n=2$ $n=3$

$\frac{-4}{3^1} \rightarrow a$ $r=\frac{-4}{3^2}$

not needed for Just Sum

$a=\frac{-4}{3}$

$r=\frac{-4}{3^2}=\frac{-4}{9}$

Convergent GST $|r|=\left|\frac{-4}{9}\right|=\frac{4}{9}<1$

$$\text{Sum}=\frac{a}{1-r}=\frac{-\frac{4}{3}}{1-\left(-\frac{4}{9}\right)}=\frac{-\frac{4}{\cancel{3}}}{\frac{13}{9}} \cdot \frac{\cancel{9}\,3}{13}=\boxed{\frac{-12}{13}}$$

$+$

Note: only pass to Absolute Series if "Helpful" Convergent, to use ACT.

More Series

See (*)

48. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^3-\ln n}{5n^3-9} \hookrightarrow \lim_{n\to\infty} \frac{2n^3-\ln n}{5n^3-9}\frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{2-\cancel{\frac{\ln n}{n^3}}^{\to 0}}{5-\cancel{9/n^3}_{\to 0}} = \frac{2}{5} \neq 0$

$\Rightarrow$ Series $\boxed{\text{Diverges}}$ by nTDT

(*) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x^3} \ {}^{\infty/\infty} = \lim_{x\to\infty} \frac{\frac{1}{x}}{3x^2} \ {}^{1/3x^2} = \lim_{x\to\infty} \cancel{\frac{1}{3x^3}}^{\to 0} = 0$

49. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \xhookrightarrow{\text{I.T.}} f(x) = \frac{\ln x}{x^2}$

PreConditions

Positive $x>1$

Continuous $x>0$

Decreasing

$f'(x) = \dfrac{x^2(1/x) - \ln x(2x)}{x^4}$

$= \dfrac{x - 2x\ln x}{x^4}$

$= \dfrac{\cancel{x}(1-2\ln x)}{x^{\cancel{4}}}$

$= \dfrac{1-2\ln x}{x^3} < 0$ when

$1-2\ln x < 0$

$1 < 2\ln x$

$\frac{1}{2} < \ln x$

$e^{\frac{1}{2}} \quad e$

$x > \sqrt{e}$ o.k.

Compute Integral

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \ln x \cdot x^{-2}\,dx$$

$u = \ln x$	$dv = x^{-2}dx$
$du = \frac{1}{x}dx$	$v = \frac{x^{-1}}{-1} = -\frac{1}{x}$

$$= \lim_{t\to\infty} \left.\frac{-\ln x}{x}\right|_1^t + \int_1^t \frac{1}{x^2}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{-\ln x}{x}\right|_1^t - \left.\frac{1}{x}\right|_1^t$$

$$= \lim_{t\to\infty} \frac{-\ln t}{t}^{\,-\infty/\infty} + \cancel{\frac{\ln 1}{1}}^{\,0} - \cancel{\frac{1}{t}}^{\,0} + 1$$

$$\overset{L'H}{=} \lim_{t\to\infty} \frac{\cancel{-1/t}^{\,\to 0}}{1} + 1 = \boxed{1} \quad \text{Integral Converges}$$

$\Rightarrow$ Series $\boxed{\text{Converges}}$ by I.T.

[Note: could also run CT if use Bound $\ln n \leq \sqrt{n}$ ..]

50. $\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{4n^2-2} \approx \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges p-Series $p = 3/2 > 1$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}+3}{4n^2-2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^2+3n^{3/2}}{4n^2-2} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1+\frac{3}{\sqrt{n}}}{4-\frac{2}{n^2}} = \frac{1}{4}$$

Finite, Non-Zero

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

51. $\sum_{n=1}^{\infty} \frac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}} \approx \sum_{n=1}^{\infty} \frac{n^{19}}{n^{20}} = \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges Harmonic p-Series $p=1$

$$\lim_{n\to\infty} \frac{\frac{n^{19}+40n^6+4n^3+19}{4+17n^5+n^{20}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{20}+40n^7+4n^4+19n}{4+17n^5+n^{20}} \cdot \frac{1/n^{20}}{1/n^{20}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{40}{n^{13}}+\frac{4}{n^{16}}+\frac{19}{n^{19}}}{\frac{4}{n^{20}}+\frac{17}{n^{15}}+1} = 1$$

Finite, Non-Zero

$\Rightarrow$ O.S. also $\boxed{\text{Diverges}}$ by LCT

52. $\sum_{n=2}^{\infty} \frac{e^n}{\ln n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{\ln n} \;\; \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{e^x}{\ln x} \;\; \frac{\infty}{\infty} \underset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{1/x} \;\; x$$

$$= \lim_{x\to\infty} xe^x = \infty \neq 0$$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^5}+\frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{5}{n^5} + \sum_{n=1}^{\infty} \frac{1}{5^n}$

Constant Multiple of Convergent p-Series $p=5>1$ is Convergent

Convergent Geometric $|r| = \frac{1}{5} < 1$

not needed [by Arithmetic of Series]

Sum of 2 Convergent Series is $\boxed{\text{Convergent}}$

54. $\displaystyle\sum_{n=1}^{\infty} \frac{1+3n^3}{n^5} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Convergent p-Series $p=2>1$

[Note: CT not helpful → bound in wrong order]

$$\lim_{n\to\infty} \frac{\frac{1+3n^3}{n^5}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2+3n^5}{n^5} = \lim_{n\to\infty} \frac{1}{n^3} + 3 = 3$$

$n^3 \to \infty$

OR $\frac{1/n^5}{1/n^5}$

Finite, Non-zero

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

positive $x>1$
Continuous $x>0$

55. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^7} \hookrightarrow f(x) = \frac{1}{x(\ln x)^7} = \left[x(\ln x)^7\right]^{-1}$ Decreasing when

$$f'(x) = -\left[x(\ln x)^7\right]^{-2}\left[\cancel{x}\cdot 7(\ln x)^6 \cdot \frac{1}{\cancel{x}} - (\ln x)^7(1)\right]$$

$$= \frac{-(\ln x)^6\left[7-\ln x\right]}{\left[x(\ln x)^7\right]^2} = \frac{-7+\ln x}{x^2(\ln x)^8} < 0$$

$\oplus$ $\oplus$

when $-7+\ln x<0 \Rightarrow \ln x > 7$
$\Rightarrow x > e^7$ ✓ o.k.

Compute

$$\int_2^{\infty} \frac{1}{x(\ln x)^7}dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^7}dx$$

$\boxed{u=\ln x \quad du = \frac{1}{x}dx}$

$$= \lim_{t\to\infty}\int_{\ln 2}^{\ln t} \frac{1}{u^7}du$$

$\boxed{x=2 \Rightarrow u=\ln 2 \quad x=t \Rightarrow u=\ln t}$

$$= \lim_{t\to\infty} \frac{-1}{6u^6}\Bigg|_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} -\frac{1}{\cancel{6(\ln t)^6}} + \frac{1}{6(\ln 2)^6} = \frac{1}{6(\ln 2)^6}$$ Integral Converges

$\Rightarrow$ o.s. $\boxed{\text{Converges}}$ by I.T.

56. $$\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2} \approx \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$ Constant Multiple of Convergent p-Series $p=2>1$ is Convergent

Bound Terms

$$\frac{\arctan n}{1+n^2} \le \frac{\pi/2}{1+n^2} \le \frac{\pi/2}{n^2}$$ and $\Rightarrow$ o.s. $\boxed{\text{Converges}}$ by CT

OR

LCT

$$\sum_{n=1}^{\infty} \frac{\arctan n}{1+n^2} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$$ Conv. p-Series $p=2>1$

$$\lim_{n\to\infty} \frac{\frac{\arctan n}{1+n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \underbrace{\cancel{\arctan n}}_{\pi/2} \cdot \frac{n^2}{1+n^2}\,\frac{1/n^2}{1/n^2} = \frac{\pi}{2} \lim_{n\to\infty} \frac{1}{1+\cancel{1/n^2}} = \frac{\pi}{2}$$

Finite, Non-Zero

$\Rightarrow$ o.s. $\boxed{\text{Converges}}$ by LCT

OR Integral Test...

57. $$\sum_{n=1}^{\infty} \frac{n^2+1}{\arctan n}$$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+1}{\arctan n} = \infty \ne 0.$$

58. $\sum_{n=1}^{\infty} \frac{2n+5}{5n^3+3n^2} \approx \sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges, p-Series $p=2>1$

$$\lim_{n\to\infty} \frac{\frac{2n+5}{5n^3+3n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^3+5n^2}{5n^3+3n^2}\,\frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{2+\cancel{5/n}^{\,\to 0}}{5+\cancel{3/n}^{\,\to 0}} = \frac{2}{5}$$

Finite

Non-zero

$\Rightarrow$ O.S. also $\boxed{\text{Converges}}$ by LCT

59. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+5n-3} \xrightarrow[\text{"Helpful"}]{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{n^2+5n-3} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges, p-Series $p=2>1$

work around

Bound Terms

$\frac{1}{n^2+5n-3} \le \frac{1}{n^2}$ and

$\Rightarrow$ A.S. also Converges by C.T.

$\xLeftarrow{\text{ACT}}$

O.S. $\boxed{\text{Converges}}$ by ACT

Remember: ACT gets a Conclusion from A.S. only helpful if A.S. Converges.

60. $\sum_{n=1}^{\infty} \frac{\pi}{\arctan(2n)}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cancel{\pi}}{\arctan(2n)} \quad \frac{2}{\cancel{\pi}} = 2 \neq 0$$

$\arctan(2n) \to \pi/2$, $2n \to \infty$

61. $\sum_{n=1}^{\infty} 3+\frac{1}{3^n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3+\cancel{\tfrac{1}{3^n}}^{\,\to 0} = 3 \neq 0.$$

$3^n \to \infty$

62. $\sum_{n=1}^{\infty} e^{1/n}$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\overset{\to 0}{1/n}} = e^0 = 1 \neq 0$$

63. $$\sum_{n=1}^{\infty} \frac{6}{n^6} + \frac{1}{(n+1)^6} = 6\sum_{n=1}^{\infty} \frac{1}{n^6} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^6}$$

Constant Multiple of Convergent p-Series $p=6>1$ is Convergent

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^6} \approx \sum_{n=1}^{\infty} \frac{1}{n^6}$$ Converges p-Series $p=6>1$

Bound Terms

$$\frac{1}{(n+1)^6} \le \frac{1}{n^6}$$ and

$\Rightarrow$ Series here Converges by CT

Original Series $\boxed{\text{Converges}}$ b/c Sum of 2 Convergent Series Converges

64. $\sum_{n=1}^{\infty} \cos^2\left(\frac{\pi n^2 + n}{n^2+7}\right)$ $\boxed{\text{Diverges}}$ by nTDT b/c

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos^2\left(\frac{\pi n^2 + n}{n^2+7} \frac{1/n^2}{1/n^2}\right)$$

$$= \lim_{n\to\infty} \cos^2\left(\frac{\pi + \overset{\to 0}{1/n}}{1 + \underset{\to 0}{7/n}}\right) = \cos^2 \pi = (-1)^2 = 1 \neq 0$$

65. $\sum_{n=1}^{\infty} (-1)^n \frac{\cos^2(\pi n^2+1)}{n^2+7} \xrightarrow[\text{"Helpful"}]{\text{A.S.}} \sum_{n=1}^{\infty} \frac{\cos^2(\pi n^2+1)}{n^2+7} \approx \sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges p-Series $p=2>1$

1st

Bound Terms

work around

$$\frac{\cos^2(\pi n^2+1)}{n^2+7} \le \frac{1}{n^2+7} \le \frac{1}{n^2} \text{ and}$$

O.S. $\boxed{\text{Converges}}$ by ACT $\xleftarrow{\text{ACT}}$ $\Rightarrow$ A.S. Converges by CT

[note: O.S. A.C. but not needed here]

Even More Series

Note: here must Analyze Absolute Series (AS) b/c Q = A.C., C.C., Diverge?

66. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5n+2} \xrightarrow[\text{must}]{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{5n+1} \le \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges Harmonic p-Series $p=1$

1st

2nd AST

① $b_n = \frac{1}{5n+2} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5n+2} = 0$

③ Terms decreasing

$b_{n+1} = \frac{1}{\underbrace{5(n+1)+2}_{5n+7}} \le \frac{1}{5n+2} = b_n$

[OR, $f(x) = \frac{1}{5x+2}$

↳ $f'(x) = \frac{-5}{(5x+2)^2} < 0$]

O.S. Converges by AST

[note: CT not helpful here $\frac{1}{5n+1} \le \frac{1}{n}$

"Smaller than Diverge" not helpful

↳ inconclusive comparison]

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{5+\frac{1}{n}} = \frac{1}{5}$$

Finite, Non-zero

$\Rightarrow$ A.S. also Diverges by LCT

↳ Not A.C.

↳ Test O.S.

O.S. $\boxed{\text{Conditionally Convergent}}$ by Definition

$\boxed{\text{C.C.}}$

67. $\sum_{n=1}^{\infty} (-1)^n \frac{n^3+6n}{n^8+1}$ $\xrightarrow[\text{must}]{\text{A.S.}}$ $\sum_{n=1}^{\infty} \frac{n^3+6n}{n^8+1} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^8} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ Converges p-Series $p=5>1$

1st

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^3+6n}{n^8+1}}{\frac{1}{n^5}} \cdot n^5 = \lim_{n\to\infty} \frac{n^8+6n^6}{n^8+1} \cdot \frac{1/n^8}{1/n^8} = \lim_{n\to\infty} \frac{1+6/n^2 \to 0}{1+1/n^8 \to 0} = 1$$

Finite, Non-zero

$\Rightarrow$ A.S. Converges by LCT

$\hookrightarrow$ O.S. $\boxed{\text{Absolutely Convergent}}$ by Definition

$\boxed{\text{A.C.}}$ Done. Not necessary to analyze O.S.

68 $\sum_{n=1}^{\infty} \frac{5^{2n}}{(2n+1)!\,\ln n}$

R.T.

$\oplus$ terms

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{5^{2(n+1)}}{[2(n+1)+1]!\,\ln(n+1)}}{\frac{5^{2n}}{(2n+1)!\,\ln n}} \right| \quad \frac{(2n+1)!\,\ln n}{5^{2n}}$$

$$= \lim_{n\to\infty} \frac{\overset{\cancel{5^{2n}}\cdot 5^2}{\cancel{5^{2n+2}}}}{\cancel{5^{2n}}} \cdot \underset{(*)}{\frac{\ln n}{\ln(n+1)}} \cdot \frac{\cancel{(2n+1)!}}{(2n+3)!}$$

$(2n+3)(2n+2)\cancel{(2n+1)!}$

$$= \lim_{n\to\infty} \frac{25}{\cancel{(2n+3)(2n+2)}} \to 0 = 0 < 1$$

$\boxed{\text{A.C.}}$

$\Rightarrow$ O.S. $\boxed{\text{Absolutely Convergent}}$ by R.T.

(*) $\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{(x+1)}} = \lim_{x\to\infty} \frac{x+1}{x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$

[or Divide $\frac{1/x}{1/x}$]

69. $\sum_{n=1}^{\infty} \frac{n!\; n^6 \cdot n^n}{10^{4n}\; e^{2n}}$ (+)

R.T.

both!

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!\,(n+1)^6\,(n+1)^{n+1}}{10^{4(n+1)}\; e^{2(n+1)}}}{\frac{n!\; n^6\, n^n}{10^{4n}\, e^{2n}}} \right| \quad \frac{10^{4n}\cdot e^{2n}}{n!\cdot n^6 \cdot n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(n+1)^6}{n^6} \cdot \frac{(n+1)^{n+1}}{n^n} \cdot \frac{10^{4n}}{10^{4n+4}} \cdot \frac{e^{2n}}{e^{2n+2}}$$

$(n+1)\,\cancel{n!}$ $\quad (n+1)^n (n+1)$ $\quad \left(\frac{n+1}{n}\,\frac{1/n}{1/n}\right)^6$ $\quad \cancel{10^{4n}} \cdot 10^4$ $\quad \cancel{e^{2n}}\, e^2$

extra $(n+1)$

explode

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)}{10^4 \cdot e^2} \cdot \frac{(n+1)^n}{n^n} \cdot \left(\frac{1+1/n}{1}\right)^6 = \infty > 1$$

$\infty \quad e \quad 1 \quad 0 \quad 1$

$\Rightarrow$ O.S. $\boxed{\text{Diverges}}$ by Ratio Test

70. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{4n+3} \xrightarrow[\text{must}]{\text{1st A.S.}} \sum_{n=1}^{\infty} \frac{1}{4n+3} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ Diverges, Harmonic p-Series $p=1$

[CT order on Bounds not helpful]

2nd AST

① $b_n = \frac{1}{4n+3} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{4n+3} = 0$

③ Terms decreasing

$$b_{n+1} = \frac{1}{\underbrace{4(n+1)+3}_{4n+7}} \leq \frac{1}{4n+3} = b_n$$

OR $f(x) = \frac{1}{4x+3}$

↳ $f'(x) = \frac{-4}{(4x+3)^2} < 0$

$\oplus$

$$\lim_{n\to\infty} \frac{\frac{1}{4n+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{4n+3}\cdot\frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{4+\cancel{3/n}} = \frac{1}{4}$$

Finite, Non-zero

$\Rightarrow$ A.S. Diverges by LCT

↳ Not A.C.

↳ Test O.S.

O.S. Converges by AST

O.S. $\boxed{\text{Conditionally Convergent}}$ by Definition

$\boxed{\text{C.C.}}$

Warning: Remember NOT to state ~~C.C. by any Test~~

71. $\sum_{n=1}^{\infty} \frac{(n!)^3 e^{2n}}{(3n)!\, n^n}$

R.T.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{[(n+1)!]^3 \cdot e^{2(n+1)}}{[3(n+1)]!\,(n+1)^{n+1}}}{\frac{(n!)^3 \cdot e^{2n}}{(3n)!\, n^n}}$$

$$\frac{(3n)! \cdot n^n}{(n!)^3 \cdot e^{2n}}$$

$$(n+1)^3 \cancel{[n!]^3} \qquad \cancel{e^{2n}} \cdot e^2$$

$$= \lim_{n\to\infty} \frac{[(n+1)!]^3}{\cancel{[n!]^3}} \cdot \frac{e^{2n+2}}{\cancel{e^{2n}}} \cdot \frac{\cancel{(3n)!}}{(3n+3)!} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$(3n+3)(3n+2)(3n+1)\cancel{(3n)!} \qquad (n+1)^n (n+1)$$

$$\frac{1}{e}$$

$$= \lim_{n\to\infty} \frac{e^{\cancel{2}}}{\cancel{e}} \cdot \left(\frac{\cancel{n+1}}{3n+3}\right)\left(\frac{n+1}{3n+2}\frac{1/n}{1/n}\right)\left(\frac{n+1}{3n+1}\frac{1/n}{1/n}\right) \cdot \frac{1}{n+1}$$

$$3\cancel{(n+1)}$$

$$= \lim_{n\to\infty} \frac{e}{3}\left(\frac{1+\cancel{1/n}}{3+\cancel{2/n}}\right)\left(\frac{1+\cancel{1/n}}{3+\cancel{1/n}}\right) \cdot \frac{1}{n+1}$$

$\frac{1}{3}$ $\quad$ $1/3$

extra $(n+1)$

makes $L \to 0$

$$= \lim_{n\to\infty} \frac{e}{3}\left(\frac{1}{\cancel{n+1}}\right) = 0 < 1$$

$$\Rightarrow \text{o.s. } \boxed{\text{A.C.}} \text{ by R.T.}$$

72. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^7+n} \xleftarrow[\text{must}]{\text{A.S.}} \sum_{n=1}^{\infty} \frac{\arctan n}{n^7+n} \approx \sum \frac{1}{n^7}$ Converges p-Series $p=7>1$

not R.T.

$$\lim_{n\to\infty} \frac{\dfrac{\arctan n}{n^7+n}}{\dfrac{1}{n^7}} = \lim_{n\to\infty} \arctan n \left[\frac{n^7}{n^7+n}\frac{1/n^7}{1/n^7}\right]$$

$$= \lim_{n\to\infty} \arctan n \left[\frac{1}{1+\cancel{1/n^6}}\right] = \pi/2$$

[OR, run CT with $\frac{\pi}{2}\sum\frac{1}{n^7}$. Const Mult of Conv. p-Series $p=7>1$ is Conv.

$$\frac{\arctan n}{n^7+n} \le \frac{\pi/2}{n^7+n} \le \frac{\pi/2}{n^7}$$]

Finite, Non-zero

$\Rightarrow$ A.S. Converges by LCT

$\Rightarrow$ O.S. $\boxed{\text{A.C.}}$ by Definition

73. $$\sum_{n=1}^{\infty} \frac{(-1)^n (3n)!\, n^2}{8^n \cdot (n!)^2 \cdot n^n}$$

R.T.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,[3(n+1)!]\,(n+1)^2}{8^{n+1}[(n+1)!]^2 (n+1)^{n+1}}}{\dfrac{\cancel{(-1)^n}(3n)!\; n^2}{8^n [n!]^2\, n^n}}\right|$$

$(3n+3)(3n+2)(3n+1)\cancel{(3n)!}$

$$= \lim_{n\to\infty} \frac{(3n+3)!}{\cancel{(3n)!}}\cdot\frac{(n+1)^2}{n^2}\cdot\frac{\cancel{8^n}}{8^{n+1}}\cdot\frac{\cancel{[n!]^2}}{[(n+1)!]^2}\cdot\frac{n^n}{(n+1)^{n+1}}$$

$\left(\frac{n+1}{n}\right)^2$ $\quad \cancel{8^n}\, 8$ $\quad (n+1)^2 \cancel{(n!)^2}$ $\quad (n+1)^n (n+1)$

$3(n+1)$

$$= \lim_{n\to\infty} \frac{\cancel{(3n+3)}(3n+2)(3n+1)}{\cancel{(n+1)}\,(n+1)\,(n+1)}\cdot\frac{1}{8}\cdot\frac{n^n}{(n+1)^n} \to \frac{1}{e}$$

$$= \lim_{n\to\infty} \frac{3}{8}\left[\frac{3+2/n}{1+1/n}\right]\cdot\left[\frac{3+1/n}{1+1/n}\right]\cdot\frac{1}{e} = \frac{27}{8e} > 1$$

note $e \approx 2.718\ldots$
$8e < 24 \Rightarrow 27 > 8e$

$\Rightarrow$ O.S. $\boxed{\text{Diverges}}$ by Ratio Test

74. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3+7}{n^7+3} \xrightarrow[\text{must}]{\text{A.S.}} \sum_{n=1}^{\infty} \frac{n^3+7}{n^7+3} \approx \sum_{n=1}^{\infty} \frac{n^3}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^4}$

Converges p-Series $p=4>1$

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^3+7}{n^7+3}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^7+7n^4}{n^7+3} \cdot \frac{1/n^7}{1/n^7} = \lim_{n\to\infty} \frac{1+\cancel{7/n^3}^{\to 0}}{1+\cancel{3/n^7}^{\to 0}} = 1$$

$\Rightarrow$ A.S. also Converges by LCT

$\Rightarrow$ O.S. $\boxed{\text{Absolutely Convergent}}$ by Definition

75. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n \cdot \pi^n (2n)!}{n^n \; 4^n \; n!}$

R.T.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} \ln(n+1)\, \pi^{n+1} [2(n+1)]!}{(n+1)^{n+1} \cdot 4^{n+1} \cdot (n+1)!}}{\frac{\cancel{(-1)^n} \cdot \ln n\, \pi^n (2n)!}{n^n \cdot 4^n \cdot n!}} \right|$$

$$= \lim_{n\to\infty} \frac{\cancel{\ln(n+1)}^{1}}{\cancel{\ln n}} \cdot \frac{\cancel{\pi^{n+1}}^{\cancel{\pi^n}\cdot\pi}}{\cancel{\pi^n}} \cdot \frac{(2n+2)!^{\,(2n+2)(2n+1)\cancel{(2n)!}}}{\cancel{(2n)!}} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n (n+1)}} \cdot \frac{\cancel{4^n}}{\underbrace{4^{n+1}}_{\cancel{4^n}\cdot 4}} \cdot \frac{\cancel{n!}}{\underbrace{(n+1)!}_{(n+1)\cancel{n!}}}$$

$(*)$

$$= \lim_{n\to\infty} \pi \cdot \frac{\overset{2(n+1)}{(2n+2)}(2n+1)\,\frac{1}{n}}{\cancel{(n+1)}\,(n+1)\,\frac{1}{n}} \cdot \underbrace{\frac{n^n}{(n+1)^n}}_{\to 1/e}$$

$$= \lim_{n\to\infty} \frac{\pi}{4e} \cdot 2\left[\frac{2+\frac{1}{n}}{1+\frac{1}{n}}\right] = \frac{\cancel{4}\pi}{\cancel{4}e} = \frac{\pi}{e} > 1$$

$\pi \approx 3.14$, $e \approx 2.718\ldots$

$\Rightarrow$ O.S. $\boxed{\text{Diverges}}$ by Ratio Test

or R.T.

(*) $$\lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \overset{\infty/\infty}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$$

OR use "Divides by" Trick