Exam #2 Fall 2021

1a. $\displaystyle\int_0^{e^4} \frac{1}{x(16+(\ln x)^2)}\,dx = \lim_{t\to 0^+} \int_t^{e^4} \frac{1}{x(16+(\ln x)^2)}\,dx$

$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}}$

$\displaystyle = \lim_{t\to 0^+} \int_{\ln t}^{4} \frac{1}{16+u^2}\,du$ u-rule

$\boxed{\begin{array}{l} x = t \Rightarrow u = \ln t \\ x = e^4 \Rightarrow u = \ln(e^4) = 4 \end{array}}$

$\displaystyle = \lim_{t\to 0^+} \frac{1}{4} \arctan\left(\frac{u}{4}\right)\Bigg|_{\ln t}^{4}$

$\displaystyle = \lim_{t\to 0^+} \frac{1}{4}\left(\arctan\left(\frac{4}{4}\right) - \arctan\left(\frac{\ln t}{4}\right)\right)$

$\pi/4$ $\quad 1 \quad$ $-\pi/2$ $\quad -\infty \quad 0^+$

$\displaystyle = \frac{1}{4}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = \frac{1}{4}\left(\frac{3\pi}{4}\right) = \frac{3\pi}{16}$ Converges

$\frac{2\pi}{4}$

1b. $\displaystyle\int_{-\infty}^{0} \frac{1}{x^2+2x+4}\,dx = \lim_{t\to -\infty} \int_t^{0} \frac{1}{x^2+2x+4}\,dx = \lim_{t\to -\infty} \int_t^{0} \frac{1}{(x+1)^2+3}\,dx$

Complete Square

Note: Discriminant

$b^2-4ac = 4-4(1)(4) = -12 < 0$

↳ Complete the Square

$\displaystyle = \lim_{t\to -\infty} \int_{t+1}^{1} \frac{1}{u^2+3}\,du$

$\boxed{\begin{array}{l} u = x+1 \\ du = dx \end{array}}$ $\boxed{\begin{array}{l} x = t \Rightarrow u = t+1 \\ x = 0 \Rightarrow u = 1 \end{array}}$

$\displaystyle = \lim_{t\to -\infty} \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{t+1}^{1}$

$\pi/6$ $\quad -\pi/2 \quad -\infty$

$\displaystyle = \lim_{t\to -\infty} \frac{1}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \arctan\left(\frac{t+1}{\sqrt{3}}\right)\right)$

$\displaystyle = \frac{1}{\sqrt{3}}\left(\frac{\pi}{6} + \frac{\pi}{2}\right) = \frac{1}{\sqrt{3}}\left(\frac{\overset{2}{\cancel{4}}\pi}{\underset{3}{\cancel{6}}}\right) = \frac{2\pi}{3\sqrt{3}}$ Converges

$\frac{3\pi}{6}$

1c. $\displaystyle\int_0^e \ln x\,dx = \lim_{t\to 0^+}\int_t^e \ln x\,dx \overset{IBP}{=} \lim_{t\to 0^+} x\ln x\Big|_t^e - \int_t^e 1\,dx$

$\boxed{\begin{array}{ll} u=\ln x & dv = 1\,dx \\ du = \frac{1}{x}dx & v = x \end{array}}$

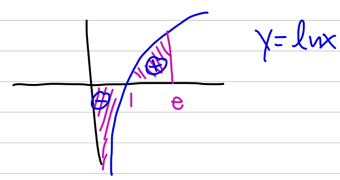
$\displaystyle = \lim_{t\to 0^+} x\ln x\Big|_t^e - x\Big|_t^e$

$\displaystyle = \lim_{t\to 0^+} e\ln e - \underbrace{t\ln t}_{0\cdot(-\infty)\to 0} - (e - t)$

See (*)

$= e - e = 0$

Makes Sense: Area Cancels



(*) $\displaystyle \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{\frac{-\infty}{\infty}}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$

1d. $\displaystyle\int_1^{\sqrt{3}} \frac{x^4 - x^3 + 3x^2 - x + 2}{x^3 - x^2 + 3x - 3}\,dx = \int_1^{\sqrt{3}} \frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+3)}\,dx$

Improper Rational Function

Improper Integral

Long Division

$\displaystyle = \lim_{t\to 1^+}\int_t^{\sqrt{3}} \frac{x^4 - x^3 + 3x^2 - x + 2}{(x-1)(x^2+3)}\,dx$

$$\begin{array}{r} x \\ x^3 - x^2 + 3x - 3 \,\big)\, x^4 - x^3 + 3x^2 - x + 2 \\ -(x^4 - x^3 + 3x^2 - 3x) \\ \hline 2x + 2 \end{array}$$

$\displaystyle = \lim_{t\to 1^+}\int_t^{\sqrt{3}} x + \frac{2x+2}{(x-1)(x^2+3)}\,dx$

$\displaystyle = \lim_{t\to 1^+}\int_t^{\sqrt{3}} x + \frac{1}{x-1} + \frac{1-x}{x^2+3}\,dx$ FREE PARTIAL FRACTIONS GIVEN

$\displaystyle = \lim_{t\to 1^+} \frac{x^2}{2} - \ln|x-1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - \frac{1}{2}\ln|x^2+3|\,\Bigg|_t^{\sqrt{3}}$

$\displaystyle = \lim_{t\to 1^+} \underbrace{\frac{3}{2}}_{\text{Finite}} - \underbrace{\ln|\sqrt{3}-1|}_{\text{Finite}} + \underbrace{\frac{1}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}}{\sqrt{3}}\right)}_{\text{Finite}} - \underbrace{\frac{1}{2}\ln 6}_{\text{Finite}} - \left(\underbrace{\frac{t^2}{2}}_{\text{Finite}} - \underbrace{\ln|t-1|}_{\text{Blows Up}} + \underbrace{\frac{1}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)}_{\text{Finite}} - \underbrace{\frac{1}{2}\ln|t^2+3|}_{\text{Finite}}\right)$

$= -\infty$ Diverges

2a. $\left\{ n \sin\left(\frac{1}{n}\right) \right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\cos(1/x)\cancel{(-1/x^2)}}{\cancel{-1/x^2}} = 1$$

$\infty \cdot 0$

$\Rightarrow$ The Sequence Converges to 1

2b. $\sum_{n=1}^{\infty} n\sin(1/n)$ Diverges by nTDT b/c

$$\lim_{n\to\infty} n\sin(1/n) = \cdots = 1 \neq 0$$

See 2a above.

3. $\sum_{n=1}^{\infty} (-1)^n \frac{\cos^2 n}{n^7+6} \xrightarrow{A.S.} \sum_{n=1}^{\infty} \frac{\cos^2 n}{n^7+6} \approx \sum_{n=1}^{\infty} \frac{1}{n^7}$ Converges, p-Series $p=7>1$

<u>Bound Terms</u>

$$\frac{\cos^2 n}{n^7+6} \le \frac{1}{n^7+6} \le \frac{1}{n^7}$$

$\Rightarrow$ A.S. Converges by C.T.

$\Rightarrow$ O.S. Converges by ACT

4a. Absolutely Convergent example

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2+9}{n^9+2} \longrightarrow \sum_{n=1}^{\infty} \frac{n^2+9}{n^9+2} \approx \sum_{n=1}^{\infty} \frac{n^2}{n^9} = \sum_{n=1}^{\infty} \frac{1}{n^7}$$ Converges p-Series $p=7>1$

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{n^2+9}{n^9+2}}{\frac{1}{n^7}} = \lim_{n\to\infty} \frac{n^9+9n^7}{n^9+2} \cdot \frac{1/n^9}{1/n^9} = \lim_{n\to\infty} \frac{1+\cancel{\frac{9}{n^2}}^{\,0}}{1+\cancel{\frac{2}{n^2}}^{\,0}} = 1$$ Finite Non-Zero

$\Rightarrow$ A.S. Converges by LCT

$\Rightarrow$ O.S. Absolutely Converges (by Definition)

OR Set-Up a Ratio Test with $L<1$

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^n (n!)^2}{(2n)!\, 7^n}$$

4b. Divergent example

$$\sum_{n=1}^{\infty} 4+\frac{1}{n}$$ Diverges by nTDT b/c $\lim_{n\to\infty} 4+\cancel{\frac{1}{n}}^{\,0} = 4 \neq 0$

OR// $$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| \frac{n!}{n^n} = \lim_{n\to\infty} \underbrace{\frac{(n+1)^{n+1}}{n^n}}_{\to e} \cdot \frac{\cancel{n!}}{(n+1)!} = e > 1$$

$(n+1)^n(\cancel{n+1})$

$(\cancel{n+1})\,\cancel{n!}$

Series Diverges by Ratio Test

OR// $$\sum_{n=1}^{\infty} \arctan n$$ Diverges by nTDT b/c $\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$

4c. Conditionally Convergent example

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{7n-1} \xrightarrow{\text{A.S.}} \sum_{n=1}^{\infty} \frac{1}{7n-1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$$ Diverges, p-Series $p=1$

AST

1. $b_n = \frac{1}{7n-1} > 0$

2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{7n-1} = 0$

3. Terms Decreasing

$$b_{n+1} = \frac{1}{\underbrace{7(n+1)-1}_{7n+6}} \le \frac{1}{7n-1} = b_n$$

Original Series Converges by A.S.T.

LCT Limit

$$\lim_{n\to\infty} \frac{\frac{1}{7n-1}}{\frac{1}{n}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{7n-1} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{7-\cancel{1/n}^{\,0}} = \frac{1}{7}$$ Finite Non-Zero

$\Rightarrow$ Absolute Series also Diverges by L.C.T.

Original Series is Conditionally Convergent by Definition

OR// $$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$$ OR// $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+6}$$

OR// $$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+5}$$ OR// $$\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+5}$$

Bonus: Study $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} \right|$$

$$= \lim_{n\to\infty} \frac{\overbrace{2^{n+1}}^{\cancel{2^n}\cdot 2}}{\cancel{2^n}} \cdot \frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}}{\cancel{n!}} \cdot \overbrace{\frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}}}^{\frac{1}{e}}$$

$$= \lim_{n\to\infty} \frac{2}{e} \cdot \cancel{\left(\frac{n+1}{n+1}\right)} = \frac{2}{e} < 1$$ Series (Absolutely) Converges by Ratio Test

Finally, because the series converges, we can conclude that

$\lim_{n\to\infty} \frac{2^n \cdot n!}{n^n} = 0$, because otherwise if the terms

don't approach 0, then the series would Diverge by nTDT

which would contradict what we proved above.