Name: Me

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121

Midterm Exam #1

February 17, 2017

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $\sinh(\ln 3)$, $e^{\ln 4}$, $\ln(e^7)$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		8
2		32
3		40
4		20
Total		100

1. [8 Points]

Use implicit differentiation to **PROVE** that $\frac{d}{dx}\arcsin(5x) = \frac{5}{\sqrt{1-25x^2}}$.

Let $y = \arcsin(5x)$

$$\sin y = 5x$$

$$\frac{d}{dx}[\sin y] = \frac{d}{dx}[5x]$$

$$\cos y \frac{dy}{dx} = 5$$

Solve $\frac{dy}{dx} = \frac{5}{\cos y} = \frac{5}{\sqrt{1-\sin^2 y}} = \frac{5}{\sqrt{1-(5x)^2}} = \frac{5}{\sqrt{1-25x^2}}$

2. [32 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

$\frac{3-3}{1-1}$ $\quad 3(1+9x^2)^{-1}$

(a) $\lim\limits_{x\to 0} \dfrac{3xe^x - \arctan(3x)}{x + \ln(1-x)} \overset{0/0}{\underset{L'H}{=}} \lim\limits_{x\to 0} \dfrac{3xe^x + 3e^x - \frac{1}{1+9x^2}(3)}{1 - \frac{1}{1-x}}$ $0/0$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{\cancel{3xe^x}^{0} + 3e^x + 3e^x \cancel{+ \frac{3}{(1+9x^2)^2}(18x)}^{0}}{-\frac{1}{(1-x)^2}}$$

$$= \frac{3+3}{-1} = \boxed{-6}$$

(b) $\lim\limits_{x\to 0} (1 + \ln(1-3x))^{\frac{1}{x}}$ $\quad 1^\infty$

$$= e^{\lim\limits_{x\to 0} \ln\left[(1+\ln(1-3x))^{1/x}\right]}$$

$$= e^{\lim\limits_{x\to 0} \frac{\ln(1+\ln(1-3x))}{x}} \quad 0/0$$

$$\underset{L'H}{=} e^{\lim\limits_{x\to 0} \frac{\frac{1}{1+\ln(1-3x)} \cdot \frac{1}{1-3x}(-3)}{1}}$$

$$= \boxed{e^{-3}}$$

2. (Continued) Evaluate the following **limit**. Please justify your answer.

(c) $\lim\limits_{x\to\infty} \left[1+\arcsin\left(\frac{1}{x}\right)+\sin\left(\frac{1}{x}\right)\right]^{x}$ $\quad 1^{\infty}$

$$= e^{\lim\limits_{x\to\infty} \ln\left[\left(1+\arcsin(1/x)+\sin(1/x)\right)^{x}\right]}$$

$$= e^{\lim\limits_{x\to\infty} x\ln\left[1+\arcsin(1/x)+\sin(1/x)\right]}$$

$$= e^{\lim\limits_{x\to\infty} \frac{\ln\left[1+\arcsin(1/x)+\sin(1/x)\right]}{1/x}} \quad \frac{0}{0}$$

$$\overset{L'H}{=} e^{\lim\limits_{x\to\infty} \frac{\frac{1}{1+\arcsin(1/x)+\sin(1/x)}\left[\frac{1}{\sqrt{1-(1/x)^2}}\cancel{\left(-\frac{1}{x^2}\right)}+\cos(1/x)\cancel{\left(-\frac{1}{x^2}\right)}\right]}{\cancel{-\frac{1}{x^2}}}}$$

$$= e^{1\cdot[1+1]}$$

$$= \boxed{e^{2}}$$

3. [40 Points] Compute the following **definite integral**. Please simplify your answer.

(a) $\int_0^1 x \arctan x \, dx = \frac{x^2}{2} \arctan x \Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\, dx$

$u = \arctan x \qquad dv = x\, dx$

$du = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}$

$= \qquad - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{1+x^2}\, dx$

$= \qquad - \frac{1}{2}\int_0^1 \frac{\cancel{x^2+1}}{\cancel{1+x^2}} - \frac{1}{1+x^2}\, dx$

$x \qquad -\arctan x$

$= \frac{x^2}{2}\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x \Big|_0^1$

$\arctan 0 = 0$

$= \frac{1}{2}\underbrace{\arctan 1}_{\pi/4} - \frac{1}{2} + \frac{1}{2}\underbrace{\arctan 1}_{\pi/4} - (0 - 0 + 0)$

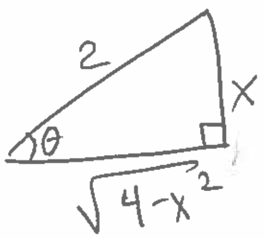
$= \boxed{\frac{\pi}{4} - \frac{1}{2}}$

3. (Continued) Compute the following **definite integral**. Please simplify your answer.

(b) $$\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{x=1}^{x=\sqrt{3}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\, d\theta$$

$$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \end{aligned}} = \int_{x=1}^{x=\sqrt{3}} \frac{4\sin^2\theta \cdot \cancel{2\cos\theta}\, d\theta}{\cancel{\sqrt{4\cos^2\theta}}}$$

$$\cancel{2\cos\theta}$$



$$= 4\int_{x=1}^{x=\sqrt{3}} \sin^2\theta \, d\theta$$

$$= 4\int_{x=1}^{x=\sqrt{3}} \frac{1-\cos(2\theta)}{2}\, d\theta$$

$$= 2\int_{x=1}^{x=\sqrt{3}} 1-\cos(2\theta)\, d\theta$$

$$= 2\left[\theta - \frac{\overbrace{\sin(2\theta)}^{2\sin\theta\cos\theta}}{\cancel{2}}\right]\Bigg|_{x=1}^{x=\sqrt{3}}$$

$$= 2\left[\arcsin\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right]\Bigg|_{x=1}^{x=\sqrt{3}}$$

$$= 2\left[\underbrace{\arcsin\frac{\sqrt{3}}{2}}_{\pi/3} - \cancel{\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{1}}{2}} - \left(\underbrace{\arcsin\frac{1}{2}}_{\pi/6} - \cancel{\frac{1}{2}\cdot\frac{\sqrt{3}}{2}}\right)\right]$$

cancel

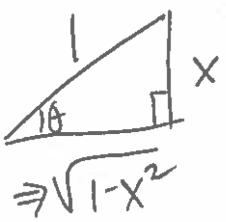
$$= 2\left[\underbrace{\frac{\pi}{3}}_{2\pi/6} - \frac{\pi}{6}\right] = 2\left[\frac{\pi}{6}\right] = \boxed{\frac{\pi}{3}}$$

3. (Continued) Compute the following **definite integral**. Please simplify your answer.

(c) $\int x^3\sqrt{1-x^2}\,dx$ using a **trigonometric substitution**.

$$\boxed{\begin{aligned}x&=\sin\theta\\ dx&=\cos\theta\, d\theta\end{aligned}}$$

$$=\int \sin^3\theta\, \underbrace{\sqrt{1-\sin^2\theta}}_{\underbrace{\sqrt{\cos^2\theta}}_{\cos\theta}}\cdot \cos\theta\, d\theta$$



$$=\int \sin^3\theta\cdot\cos^2\theta\, d\theta$$

$$=\int \sin^2\theta\cdot\cos^2\theta\cdot\sin\theta$$

$$=\int (1-\cos^2\theta)\cos^2\theta\cdot\sin\theta$$

$$\boxed{\begin{aligned}u&=\cos\theta\\ du&=-\sin\theta\, d\theta\\ -du&=\sin\theta\, d\theta\end{aligned}}$$

$$=-\int (1-u^2)u^2\, du$$

$$=-\int u^2-u^4\, du$$

$$=-\left[\frac{u^3}{3}-\frac{u^5}{5}\right]+C$$

$$=-\frac{\cos^3\theta}{3}+\frac{\cos^5\theta}{5}+C$$

$$=\boxed{-\frac{\left(\sqrt{1-x^2}\right)^3}{3}+\frac{\left(\sqrt{1-x^2}\right)^5}{5}+C}$$

OR $\boxed{-\frac{1}{3}(1-x^2)^{3/2}+\frac{1}{5}(1-x^2)^{5/2}+C}$

3. (Continued) Compute the following **definite integral**. Please simplify your answer.

(d) $\int_1^{e^3} (\ln x)^2 \, dx$

$= x(\ln x)^2 \Big|_1^{e^3} - 2\int_1^{e^3} \ln x \, dx$

$\cancel{u = (\ln x)^2}$ $dv = 1dx$

$du = 2\ln x \left(\frac{1}{x}\right)$ $v = x$

cancel

$= x(\ln x)^2 \Big|_1^{e^3} - 2\left[x \ln x \Big|_1^{e^3} - \int_1^{e^3} 1 \, dx\right]$

$u = \ln x$ $dv = 1dx$

$du = 1/x$ $v = x$

$= x(\ln x)^2 \Big|_1^{e^3} - 2x\ln x \Big|_1^{e^3} + 2x \Big|_1^{e^3}$

$= e^3 \underbrace{(\ln e^3)}_{3}{}^2 - \overset{\ln 1 = 0}{0} - (2e^3 \overset{3}{\cancel{\ln e^3}} - 0) + 2e^3 - 2$

$= 9e^3 - 6e^3 + 2e^3 - 2$

$= \boxed{5e^3 - 2}$

4. [20 Points] Compute the following **indefinite integral.**

(a) $\displaystyle\int \frac{1}{(1+x^2)\,[5+(\arctan x)^2]}\,dx = \int \frac{1}{5+w^2}\,dw$

$w = \arctan x$

$dw = \dfrac{1}{1+x^2}\,dx$

$= \dfrac{1}{\sqrt{5}} \arctan\left(\dfrac{w}{\sqrt{5}}\right) + C$

$= \boxed{\dfrac{1}{\sqrt{5}} \arctan\left(\dfrac{\arctan x}{\sqrt{5}}\right) + C}$

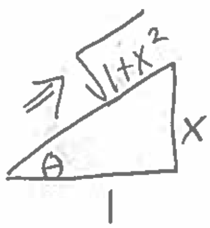
4. (Continued) Compute the following **indefinite integral**.

(b) $\displaystyle\int \frac{1}{(1+x^2)^{\frac{5}{2}}}\,dx = \int \frac{1}{\left(\sqrt{1+\tan^2\theta}\right)^5}\cdot \sec^2\theta\, d\theta$

$$\boxed{\begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta\, d\theta \end{aligned}}$$

$$= \int \frac{1}{\left(\sqrt{\sec^2\theta}\right)^5}\sec^2\theta\, d\theta$$

$\hookrightarrow \sec^5\theta$



$$= \int \frac{1}{\sec^3\theta}\, d\theta$$

$$= \int \cos^3\theta\, d\theta$$

$$= \int \cos^2\theta \cdot \cos\theta\, d\theta$$

$$= \int (1-\sin^2\theta)\cos\theta\, d\theta$$

$$\boxed{\begin{aligned} w &= \sin\theta \\ dw &= \cos\theta\, d\theta \end{aligned}}$$

$$= \int 1 - w^2\, dw$$

$$= w - \frac{w^3}{3} + C$$

$$= \sin\theta - \frac{\sin^3\theta}{3} + C$$

$$= \boxed{\frac{x}{\sqrt{1+x^2}} - \frac{1}{3}\left(\frac{x}{\sqrt{1+x^2}}\right)^3 + C}$$

****

OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

****

OPTIONAL BONUS #1 Compute the following **indefinite integral.**

1. $\int \sec^3 x\, dx = \int \sec^2 x \cdot \sec x\, dx = \sec x \tan x - \int \sec x \underbrace{\tan^2 x}_{\sec^2 x - 1}\, dx$

$$\boxed{\begin{array}{ll} u = \sec x & dv = \sec^2 x\, dx \\ du = \sec x \tan x\, dx & v = \tan x \end{array}}$$

$= \sec x \tan x - \int \sec^3 x - \sec x\, dx$

$= \sec x \tan x - \int \sec^3 x + \underbrace{\int \sec x\, dx}_{\ln|\sec x + \tan x|}$

Rewrite:

$\int \sec^3 x\, dx = \sec x \tan x - \int \sec^3 x\, dx + \ln|\sec x + \tan x|$

$\Rightarrow 2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x \tan x| \Rightarrow \int \sec^3 x\, dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + c$

OPTIONAL BONUS #2 Compute the following **indefinite integral.**

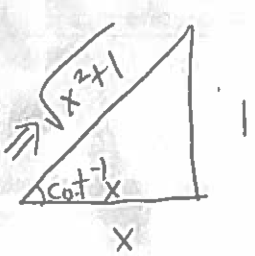
2. $\int \frac{1}{1 + 3\sin^2 x}\, dx = \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{3\sin^2 x}{\cos^2 x}}\, dx = \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x}\, dx = \int \frac{\sec^2 x}{(1 + \tan^2 x) + 3\tan^2 x}\, dx$

$= \int \frac{\sec^2 x}{1 + 4\tan^2 x}\, dx = \int \frac{\sec^2 x}{1 + (2\tan x)^2}\, dx = \frac{1}{2}\int \frac{1}{1 + w^2}\, dw = \frac{1}{2}\arctan w + C$

$$\boxed{\begin{array}{l} w = 2\tan x \\ dw = 2\sec^2 x\, dx \\ \frac{1}{2}dw = \sec^2 x\, dx \end{array}}$$

$= \frac{1}{2}\arctan(2\tan x) + C$

OPTIONAL BONUS #3 Show that $\cos(\arctan(\sin(\cot^{-1} x))) = \sqrt{\frac{x^2 + 1}{x^2 + 2}}$



$\sin(\cot^{-1} x) = \frac{1}{\sqrt{x^2+1}} \Rightarrow \cos(\arctan(\sin(\cot^{-1} x))) = \frac{\sqrt{x^2+1}}{\sqrt{x^2+2}}$