Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 121 Final Exam

December 19, 2019

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

- Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		12
2		18
3		40
4		18
5		24
6		20
7		10
8		10
9		10
10		18
11		20
Total		200

1. [12 Points] Evaluate the following **limits**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. Simplify.

(a) $\displaystyle \lim_{x\to 0} \frac{xe^x - \sin x}{\ln(1+x) - \arctan x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{xe^x + e^x - \cos x}{\frac{1}{1+x} - \frac{1}{1+x^2}} = \lim_{x\to 0} \frac{xe^x+e^x-\cos x}{(1+x)^{-1} - (1+x^2)^{-1}}$ rewrite

$$\underset{L'H}{=} \lim_{x\to 0} \frac{xe^x + e^x + e^x + \sin x}{-(1+x)^{-2} + (1+x^2)^{-2}(2x)} = \lim_{x\to 0} \frac{\cancel{xe^x} + 2e^x + \cancel{\sin x}}{\frac{-1}{(1+x)^2} + \frac{2x}{(1+x^2)^2}} = \frac{2}{-1} = \boxed{-2}$$

(b) Compute $\displaystyle \lim_{x\to 0} \frac{xe^x - \sin x}{\ln(1+x) - \arctan x}$ **again** using series.

$$= \lim_{x\to 0} \frac{x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{x} + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots - \cancel{x} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots}{\cancel{x} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots - \cancel{x} + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots}$$

$$= \lim_{x\to 0} \frac{x^2 + \frac{x^3}{2!} + \frac{x^3}{3!} + \frac{x^4}{3!} + \frac{x^5}{4!} - \frac{x^5}{5!} + \cdots}{-\frac{x^2}{2} + \frac{2x^3}{3} - \frac{x^4}{4} - \cdots} \cdot \frac{\left(1/x^2\right)}{\left(1/x^2\right)}$$

$$= \lim_{x\to 0} \frac{1 + \frac{x}{2!} + \frac{x}{3!} + \frac{x^2}{3!} + \cdots}{-\frac{1}{2} + \frac{2x}{3} - \frac{x^2}{4} + \cdots} = \frac{1}{\left(-\frac{1}{2}\right)} = \boxed{-2}$$

all x terms $\to 0$

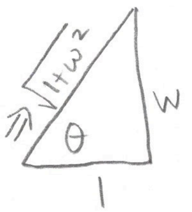
Match!

2. [18 Points] Evaluate the following **integral**.

(a) Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sin^2 x)^{\frac{7}{2}}}\, dx = \frac{43}{60\sqrt{2}}$

$$\boxed{\begin{matrix} w = \sin x \\ dw = \cos x\, dx \end{matrix}} \quad = \int_{x=0}^{x=\pi/2} \frac{1}{\left(\sqrt{1+w^2}\right)^7}\, dw = \int_{x=0}^{x=\pi/2} \frac{1}{\Big(\underbrace{\sqrt{\underbrace{1+\tan^2\theta}_{\sec^2\theta}}}_{(\sec\theta)}\Big)^7} \cdot \sec^2\theta\, d\theta$$

$$\boxed{\begin{matrix} w = \tan\theta \\ dw = \sec^2\theta\, d\theta \end{matrix}} \quad = \int_{x=0}^{x=\pi/2} \frac{1}{\sec^5\theta}\, d\theta = \int_{x=0}^{x=\pi/2} \cos^5\theta\, d\theta$$



$$= \int_{x=0}^{x=\pi/2} \cos^4\theta \cdot \cos\theta\, d\theta = \int_{x=0}^{x=\pi/2} (\underbrace{\cos^2\theta}_{1-\sin^2\theta})^2 \cos\theta\, d\theta$$

$$= \int_{x=0}^{x=\pi/2} (1-\sin^2\theta)^2 \cos\theta\, d\theta = \int_{x=0}^{x=\pi/2} (1-u^2)^2\, du$$

$$\boxed{\begin{matrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{matrix}} \quad = \int_{x=0}^{x=\pi/2} 1-2u^2+u^4\, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} \Bigg|_{x=0}^{x=\pi/2}$$

$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta \Bigg|_{x=0}^{x=\pi/2}$$

$$= \frac{w}{\sqrt{1+w^2}} - \frac{2}{3}\left(\frac{w}{\sqrt{1+w^2}}\right)^3 + \frac{1}{5}\left(\frac{w}{\sqrt{1+w^2}}\right)^5 \Bigg|_{x=0}^{x=\pi/2}$$

$$= \frac{\sin x}{\sqrt{1+\sin^2 x}} - \frac{2}{3}\left(\frac{\sin x}{\sqrt{1+\sin^2 x}}\right)^3 + \frac{1}{5}\left(\frac{\sin x}{\sqrt{1+\sin^2 x}}\right)^5 \Bigg|_0^{\pi/2}$$

$\sin 0 = 0$

$$= \frac{\overset{1}{\cancel{\sin \pi/2}}}{\sqrt{1+\underset{1}{\cancel{\sin^2 \pi/2}}}} - \frac{2}{3}\left(\frac{\overset{1}{\cancel{\sin \pi/2}}}{\sqrt{1+\underset{1}{\cancel{\sin^2 \pi/2}}}}\right)^3 + \frac{1}{5}\left(\frac{\overset{1}{\cancel{\sin \pi/2}}}{\sqrt{1+\underset{1}{\cancel{\sin^2 \pi/2}}}}\right)^5 - (0 - 0 + 0)$$

MATCH!

$$= \frac{1}{\sqrt{2}} - \frac{2}{3(\sqrt{2})^3} + \frac{1}{5(\sqrt{2})^5} = \frac{1}{\sqrt{2}} - \frac{\cancel{2}}{3\cdot\cancel{2}\sqrt{2}} + \frac{1}{5\cdot 4\sqrt{2}} = \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} = \frac{60-20+3}{60\sqrt{2}} = \boxed{\frac{43}{60\sqrt{2}}}$$

2. (Continued) Evaluate the following **integral**.

(b) $\displaystyle\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{x=1}^{x=\sqrt{3}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \cdot \cancel{2\cos\theta}\,d\theta = 4\int_{x=1}^{x=\sqrt{3}} \sin^2\theta\, d\theta$

$\sqrt{4(1-\sin^2\theta)}$

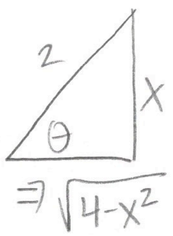
$\sqrt{4\cos^2\theta}$

$\cancel{2\cos\theta}$

$\boxed{\begin{aligned} x &= 2\sin\theta \\ dx &= 2\cos\theta\, d\theta \end{aligned}}$

$\sin\theta = \frac{x}{2}$

$\theta = \arcsin\left(\frac{x}{2}\right)$



$\displaystyle = \overset{2}{\cancel{4}}\int_{x=1}^{x=\sqrt{3}} \frac{1-\cos(2\theta)}{\cancel{2}}\,d\theta = 2\int_{x=1}^{x=\sqrt{3}} 1-\cos(2\theta)\,d\theta$

$\displaystyle = 2\left[\theta - \frac{\overset{\cancel{2}\sin\theta\cos\theta}{\sin(2\theta)}}{\cancel{2}}\right]\Bigg|_{x=1}^{x=\sqrt{3}} = 2\left[\arcsin\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right]\Bigg|_1^{\sqrt{3}}$

$\displaystyle = 2\left[\overset{\pi/3}{\cancel{\arcsin\frac{\sqrt{3}}{2}}} - \cancel{\frac{\sqrt{3}}{2}\cdot\frac{1}{2}} - \left(\overset{\pi/6}{\cancel{\arcsin\left(\frac{1}{2}\right)}} - \cancel{\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}\right)\right]$

$\displaystyle = 2\left[\underset{2\pi/6}{\frac{\pi}{3}} - \frac{\pi}{6}\right] = 2\left[\frac{\pi}{6}\right] = \boxed{\frac{\pi}{3}}$

All Limits Set-Up FIRST

3. [40 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\int_0^{\boxed{5}} \frac{6}{x^2-4x-5}\,dx = \int_0^5 \frac{6}{(x-5)(x+1)}\,dx = \lim_{t\to 5^-} \int_0^t \frac{6}{(x-5)(x+1)}\,dx$

PFD

$(x-5)(x+1)\left[\frac{6}{\cancel{(x-5)}\cancel{(x+1)}} = \frac{A}{x-5} + \frac{B}{x+1}\right](x-5)(x+1)$

$6 = A(x+1) + B(x-5)$

$= Ax + A + Bx - 5B$

$= (A+B)x + A - 5B$

- $A+B=0 \rightarrow B=-A$
- $A-5B=6$

$A-5(-A)=6$

$6A=6 \Rightarrow A=1 \Rightarrow B=-1$

$= \lim_{t\to 5^-} \int_0^t \frac{1}{x-5} - \frac{1}{x+1}\,dx$

$= \lim_{t\to 5^-} \ln|x-5| - \ln|x+1| \Big|_0^t$

$= \lim_{t\to 5^-} \ln|t-5| - \ln|t+1| - (\ln 5 - \cancel{\ln 1}^{\,0})$

5^-, 0^+, $-\infty$, 10^-, $-\ln 6$

all Finite here.

$= \boxed{-\infty}$ Diverges

(b) $\int_0^{e^5} \frac{1}{x[25+(\ln x)^2]}\,dx = \lim_{t\to 0^+} \int_t^{e^5} \frac{1}{x[25+(\ln x)^2]}\,dx = \lim_{t\to 0^+} \int_{\ln t}^{5} \frac{1}{25+w^2}\,dw$ "a-rule"

$\boxed{\begin{array}{l} w=\ln x \\ dw = \frac{1}{x}dx \end{array}}$

$\boxed{\begin{array}{l} x=t \Rightarrow w=\ln t \\ x=e^5 \Rightarrow w=\cancel{\ln e}^{5}=5 \end{array}}$

$= \lim_{t\to 0^+} \frac{1}{5}\arctan\left(\frac{w}{5}\right)\Big|_{\ln t}^{5}$

$= \lim_{t\to 0^+} \frac{1}{5}\left[\arctan\left(\frac{5}{5}\right) - \arctan\left(\frac{\ln t}{5}\right)\right]$

$\pi/4$, 1, $-\pi/2$, $-\infty$, 0^+

$= \frac{1}{5}\left[\frac{\pi}{4} + \frac{\pi}{2}\right] = \frac{1}{5}\left(\frac{3\pi}{4}\right) = \boxed{\frac{3\pi}{20}}$ Converges

x^2-4x+4

3. (Continued) For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(c) $\int_{-\infty}^{5} \frac{6}{x^2-4x+7}\,dx = \lim_{t\to-\infty} \int_t^5 \frac{6}{x^2-4x+7}\,dx = \lim_{t\to-\infty} \int_t^5 \frac{6}{(x-2)^2+3}\,dx$

b^2-4ac

$16-4(1)(7)=-12$

Quadratic Irreducible

$= \lim_{t\to-\infty} \int_{t-2}^{3} \frac{6}{w^2+3}\,dw = \lim_{t\to-\infty} \frac{6}{\sqrt{3}} \arctan\left(\frac{w}{\sqrt{3}}\right)\Big|_{t-2}^{3}$

$\boxed{\begin{array}{l} w=x-2 \\ dw=dx \end{array}}$

$\boxed{\begin{array}{l} x=t \Rightarrow w=t-2 \\ x=5 \Rightarrow w=3 \end{array}}$

$= \lim_{t\to-\infty} \frac{6}{\sqrt{3}} \left[\underbrace{\arctan\left(\frac{3}{\sqrt{3}}\right)}_{\pi/3,\ \sqrt{3}} - \underbrace{\arctan\left(\frac{t-2}{\sqrt{3}}\right)}_{-\frac{\pi}{2},\ -\infty}\right]$

$= \frac{6}{\sqrt{3}}\left[\frac{\pi}{3}+\frac{\pi}{2}\right] = \frac{6}{\sqrt{3}}\left[\frac{5\pi}{6}\right] = \boxed{\frac{5\pi}{\sqrt{3}}}$ Converges

$2\pi/6 + 3\pi/6$

(d) $\int_1^2 \frac{1}{x\ln x}\,dx = \lim_{t\to1^+} \int_t^2 \frac{1}{x\ln x}\,dx = \lim_{t\to1^+} \int_{\ln t}^{\ln 2} \frac{1}{w}\,dw$

$\boxed{\begin{array}{l} w=\ln x \\ dw=\frac{1}{x}dx \end{array}}$

$\boxed{\begin{array}{l} x=t \Rightarrow w=\ln t \\ x=2 \Rightarrow w=\ln 2 \end{array}}$

$= \lim_{t\to1^+} \ln|w|\Big|_{\ln t}^{\ln 2} = \lim_{t\to1^+} \underbrace{\ln|\ln 2|}_{\text{Finite}} - \underbrace{\ln|\ln t|}_{-\infty,\ 0^+,\ 1}$

$= \boxed{+\infty}$ Diverges

3. (Continued) For the following **improper integral**, determine whether it converges or diverges. If it converges, find its value. Simplify.

$$\frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

(e) $\displaystyle\int_0^e \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to0^+} \int_t^e \ln x \cdot x^{-1/2}\,dx = \lim_{t\to0^+} 2\sqrt{x}\ln x\Big|_t^e - 2\int_t^e x^{-1/2}\,dx$

$$\frac{x^{1/2}}{1/2}$$

IBP

$$\boxed{\begin{array}{ll} u=\ln x & dv = x^{-1/2}dx \\ du=\frac{1}{x}dx & v=2\sqrt{x} \end{array}}$$

$$= \lim_{t\to0^+} 2\sqrt{x}\ln x\Big|_t^e - 4\sqrt{x}\Big|_t^e$$

$$= \lim_{t\to0^+} 2\sqrt{e}\overset{1}{\cancel{\ln e}} - \overset{0\cdot(-\infty)}{\cancel{2\sqrt{t}\ln t}} - 4(\sqrt{e} - \overset{\to 0}{\cancel{\sqrt{t}}})$$

$(*)$

0

$$= 2\sqrt{e} - 4\sqrt{e} = \boxed{-2\sqrt{e}} \quad \text{Converges.}$$

$$(*)\ \lim_{t\to0^+} \overset{0\cdot(-\infty)}{\sqrt{t}\ln t} = \lim_{t\to0^+} \frac{\ln t}{\frac{1}{\sqrt{t}}} \overset{-\infty/\infty}{\underset{L'H}{=}} \lim_{t\to0^+} \frac{\frac{1}{t}}{\frac{-1}{2t^{3/2}}} = \lim_{t\to0^+} -2\overset{\to0}{\cancel{\sqrt{t}}} = 0$$

Flip

$$-2t^{3/2}$$

$$t^{-1/2} \to -\tfrac{1}{2}t^{-3/2}$$

4. [18 Points] Find the **sum** of each of the following series (which do converge). Simplify.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^n - 2}{4^n}$ (Hint: split?) $= \sum_{n=1}^{\infty}\left(\frac{-3}{4}\right)^n - 2\sum_{n=1}^{\infty}\frac{1}{4^n}$

$a = -3/4$ $\quad = -3/4 + (-3/4)^2 + \cdots$ $\quad$ hold $\quad 2\left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right]$ $\quad a = 1/2$ $\quad r = 1/4$ $\quad$ Sum $= \frac{a}{1-r}$

$r = -3/4$

$\text{Sum} = \frac{a}{1-r} = \frac{-3/4}{1-(-3/4)} = \frac{-3/4}{7/4}\ {}^{4/7} = -3/7$ $\quad$ Total Sum $= -\frac{3}{7} - \frac{2}{3} = \frac{-9}{21} - \frac{14}{21} = \boxed{\frac{-23}{21}}$ $\quad = \frac{\frac{1}{2}}{1-1/4}\ {}^{4/3}$, $\frac{1}{3/4}$ $= \frac{2}{3}$.

extra (-1)

(b) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(\ln 9)^n}{2^{n+1}\cdot n!} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(\ln 9)^n}{2^n\, n!} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{\ln 9}{2}\right)^n}{n!} = -\frac{1}{2}e^{-\frac{\ln 9}{2}}$

extra 2

$= -\frac{1}{2}\cancel{e}^{\cancel{\ln}\left[9^{-1/2}\right]} = -\frac{1}{2}\cdot\frac{1}{\sqrt{9}_{3}} = \boxed{\frac{-1}{6}}$

extra (-1)

(c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n-1}}{9^n\,(2n)!}\left(\frac{\pi}{\pi}\right) = -\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{3^{2n}(2n)!}\cdot\frac{1}{\pi} = \frac{-1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{3}\right)^{2n}}{(2n)!} = -\frac{1}{\pi}\cancel{\cos}\left(\frac{\pi}{3}\right)^{\frac{1}{2}} = \boxed{\frac{-1}{2\pi}}$

(d) $\frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \frac{\pi^7}{7!} - \frac{\pi^9}{9!} + \ldots = -\cancel{\sin\pi}^{\,0} + \pi = \boxed{\pi}$ $\quad$ or $-\left[\cancel{\sin\pi}^{\,0} - \pi\right] = \pi$

$\sin\pi = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \cdots \rightarrow -\sin\pi = -\pi + \frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \cdots$

(e) $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right) = -\ln(1+1) = \boxed{-\ln 2}$

(f) $3 - 1 + \frac{3}{5} - \frac{3}{7} + \frac{3}{9} - \ldots = 3 - \frac{3}{3} + \frac{3}{5} - \frac{3}{7} + \frac{3}{9} - \ldots = 3\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right]$

$= 3\cancel{\arctan(1)}^{\,\pi/4} = \boxed{\frac{3\pi}{4}}$

5. [24 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+7}{n^7+2}$

A.S. $\sum_{n=1}^{\infty} \frac{n^2+7}{n^7+2} \underset{N}{\sim} \sum_{n=1}^{\infty} \frac{n^2}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$ Converges p-Series $p=5>1$

$$\lim_{n\to\infty} \frac{\frac{n^2+7}{n^7+2}}{\frac{1}{n^5}} = \lim_{n\to\infty} \frac{n^7+7n^5}{n^7+2} = 1 \quad \text{Finite, Non-zero.}$$

don't need to justify for Finals (for me)

$\Rightarrow$ A.S. also Converges by LCT

$\Rightarrow$ O.S. $\boxed{\text{A.C.}}$ by definition.

(b) $\sum_{n=1}^{\infty} \frac{\arctan n}{7} + \frac{7}{\arctan n}$ Diverges by nTDT b/c

$$\lim_{n\to\infty} \frac{\arctan n}{7} + \frac{7}{\arctan n} = \lim_{n\to\infty} \frac{\pi/2}{7} + \frac{7}{\pi/2} = \frac{\pi}{14} + \frac{14}{\pi} \neq 0.$$

5. (Continued) In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(c) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{\sqrt{n}+7}{n}\right)$ $\xrightarrow{A.S.}$ $\sum_{n=1}^{\infty} \frac{\sqrt{n}+7}{n} \approx \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ Diverges p-Series $p = 1/2 < 1$

CT

Bound Terms

$\frac{\sqrt{n}+7}{n} \geq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ $\Rightarrow$ A.S. also Diverges by CT

[LCT works also]

AST

O.S.

① $b_n = \frac{\sqrt{n}+7}{n} > 0$

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}+7}{n} = 0$

③ Terms Decreasing

$f(x) = \frac{\sqrt{x}+7}{x}$ has $f'(x) = \frac{x\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}+7)(1)}{x^2}$

$= \frac{\frac{\sqrt{x}}{2} - \sqrt{x} - 7}{x^2}$

$-\left(\frac{\sqrt{x}}{2}+7\right)$ $\ominus$

$= \frac{-\frac{\sqrt{x}}{2} - 7}{x^2} \ominus < 0$

$\oplus$

O.S. Converges by A.S.T.

+

O.S. $\boxed{C.C.}$ by Definition

WARNING: Remember ~~not to declare~~ C.C. by A.S.T. why?

5. (Continued) Determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3n)!\,\ln n}{(n!)^2\,e^{4n}\,n^n}$

Ratio Test

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,[3(n+1)]!\,\ln(n+1)}{[(n+1)!]^2\,e^{4(n+1)}\,(n+1)^{n+1}}}{\dfrac{\cancel{(-1)^n}\,(3n)!\,\ln n}{(n!)^2\,e^{4n}\,n^n}}\right|$$

$$= \lim_{n\to\infty} \frac{\overset{(3n+3)(3n+2)(3n+1)(3n)!}{(3n+3)!}}{(3n)!}\cdot \underbrace{\cancel{\frac{\ln(n+1)}{\ln n}}}_{(*)}\cdot \frac{\cancel{(n!)^2}}{\underset{(n+1)^2(\cancel{n!})^2}{[(n+1)!]^2}}\cdot\frac{\cancel{e^{4n}}}{\underset{\cancel{e^{4n}}\cdot e^4}{e^{4n+4}}}\cdot\frac{n^n}{\underset{(n+1)^n(n+1)}{(n+1)^{n+1}}}$$

$$=\lim_{n\to\infty}\left(\frac{\overset{3(\cancel{n+1})}{3n+3}}{\cancel{n+1}}\right)\left(\frac{3n+2}{n+1}\right)\left(\frac{3n+1}{n+1}\right)\cdot\frac{1}{e^4}\cdot\frac{n^n}{(n+1)^n}$$

$\frac{3n+2}{n+1}\to 3$, $\frac{3n+1}{n+1}\to 3$, $\frac{n^n}{(n+1)^n}\to\frac{1}{e}$

$$=\frac{27}{e^5}<1 \qquad \text{O.S. } \boxed{\text{A.C}} \text{ by R.T.}$$

$e^5 > 2^5 = 32$

$$(*)\ \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{x}{x+1} \underset{\text{L'H}}{=} \lim_{x\to\infty}\frac{1}{1} = 1$$

6. [20 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x-5)^n}{(n+7)^2 \cdot 7^{n+1}}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(3x-5)^{n+1}}{(n+8)^2\,7^{n+2}}}{\frac{\cancel{(-1)^n}(3x-5)^n}{(n+7)^2\,7^{n+1}}}\right| = \lim_{n\to\infty}\underbrace{\left|\frac{(3x-5)^{n+1}}{(3x-5)^n}\right|}_{|3x-5|}\cdot\left[\frac{\cancel{n+7}}{\cancel{n}+8}\right]^2_{\to 1}\cdot\underbrace{\frac{\cancel{7^{n+1}}}{\cancel{7^{n+2}}}}_{1/7}$$

$$= \frac{|3x-5|}{7} < 1$$

Converges by R.T. when

$|3x-5| < 7$ $\quad -7 < 3x-5 < 7$

$+5 \quad +5 \quad +5$

$-2 < 3x < 12$

$\div 3$

$-\frac{2}{3} < x < 4$

Manually Check Endpoints

Take $x = 4$ O.S. becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n\,7^n}{(n+7)^2\,7^{n+1}} = \frac{1}{7}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+7)^2} \xrightarrow{\text{A.S.}} \frac{1}{7}\sum_{n=1}^{\infty}\frac{1}{(n+7)^2} \sim \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Converges p-Series $p = 2 > 1$

[OR LCT]

~~OR~~ AST here

CT Bound Terms

$$\frac{1}{7}\frac{1}{(n+7)^2} \le \frac{1}{(n+7)^2} \le \frac{1}{n^2}$$

$\Rightarrow$ A.S Converges by CT

$\Rightarrow$ O.S. Converges by ACT.

Take $x = -\frac{2}{3}$ O.S. becomes

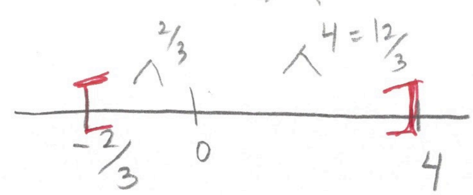
$\cancel{(-1)^{2n}}$ 1 even

$(-1)^n(-1)^n 7^n$ > justify cancel

$$\sum_{n=1}^{\infty}\frac{(-1)^n\left[3\left(-\frac{2}{3}\right)-5\right]^n}{(n+7)^2\cdot 7^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^n(-7)^n}{(n+7)^2\,7^{n+1}} = \sum_{n=1}^{\infty}\frac{7^n}{(n+7)^2\,7^{n+1}} = \underbrace{\frac{1}{7}\sum_{n=1}^{\infty}\frac{1}{(n+7)^2}}$$

Converges as shown (already) above.

$\frac{14}{3}$ Half $\to \frac{7}{3}$



$$\boxed{\begin{array}{l} I = \left[-\frac{2}{3}, 4\right] \\ R = \frac{7}{3} \end{array}}$$

6. (Continued) Find the **Interval** and **Radius** of Convergence for each of the following power series. Analyze carefully and with full justification.

(b) $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n^n}$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{2(n+1)+1}}{(n+1)^{n+1}}}{\frac{x^{2n+1}}{n^n}}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{x^{2n+1}}\right| \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n(n+1)}}$$

$$= \lim_{n\to\infty} |x|^2 \underbrace{\frac{n^n}{(n+1)^n}}_{\to 1/e} \cdot \underbrace{\frac{1}{n+1}}_{\to 0} = 0 < 1$$ Converges by R.T. for all x

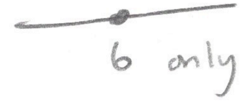
(always no matter what x is)

$$\boxed{\begin{array}{l} I = (-\infty, \infty) \\ R = \infty \end{array}}$$

(c) $\sum_{n=1}^{\infty} n!\,(x-6)^n$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}(x-6)^{n+1}}{\cancel{n!}\,(x-6)^n}\right| = \lim_{n\to\infty} (n+1)|x-6| = \infty > 1$$ Diverges by R.T. unless $x = 6$

$$\boxed{\begin{array}{l} I = \{6\} \\ R = 0 \end{array}}$$



$e^x = 1 + x + x^2/2! + x^3/3! + \cdots$

$$\begin{array}{r} 24 \\ 16 \\ \hline 144 \\ 240 \\ \hline 384 \end{array}$$

7. [10 Points] Please analyze with detail and justify carefully. Simplify.

(a) Use MacLaurin Series to **Estimate** $\dfrac{1}{\sqrt{e}}$ with error less than $\dfrac{1}{100}$.

$$\frac{1}{\sqrt{e}} = e^{-1/2} = 1 + \left(-\tfrac{1}{2}\right) + \frac{\left(-\frac{1}{2}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)^3}{3!} + \frac{\left(-\frac{1}{2}\right)^4}{4!} - \cdots$$

$$= 1 - \tfrac{1}{2} + \frac{\left(\frac{1}{4}\right)}{2} - \frac{\left(\frac{1}{8}\right)}{6} + \frac{\left(\frac{1}{16}\right)}{24} - \cdots$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{8} - \tfrac{1}{48} + \tfrac{1}{384} - \cdots$$

$$\approx 1 - \tfrac{1}{2} + \tfrac{1}{8} - \tfrac{1}{48} = \frac{48-24+6-1}{48} = \boxed{\frac{29}{49}} \leftarrow \text{Estimate.}$$

Using ASET the error is at most $\frac{1}{384} < \frac{1}{100}$ as desired.

(b) Compute the MacLaurin Series for $f(x) = \dfrac{1}{(1-x)^2}$ and then **State** the Radius of Convergence. Your answer should be in Sigma notation. Hint: Use Differentiation.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{+1}{1-x}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \boxed{\sum_{n=0}^{\infty} n x^{n-1}}$$

Check chain rule: $(1-x)^{-1} \to -(1-x)^{-2}(-1)$ $+$

$R=1$

Conv. $|x|<1$ GST

$R=1$ STILL.

OR $\displaystyle\sum_{n=1}^{\infty} n x^{n-1}$

8. [10 Points] For both parts, you do **not** need to ~~find the Radius~~ of Convergence. Your answer should be in Sigma notation or write out the first 5 non-zero terms.

$a=0$

(a) Demonstrate one method to compute the MacLaurin Series for $\boldsymbol{f(x) = \ln(1+x)}$. Justify. Do not just write down the formula.

Chart Method

$f(x) = \ln(1+x)$ $\quad f(0) = \ln 1 = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ $\quad f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ $\quad f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ $\quad f'''(0) = 2$

$f^{(4)}(x) = -6(1+x)^{-4}$ $\quad f^{(4)}(0) = -6$

$f^{(5)}(x) = 24(1+x)^{-5}$ $\quad f^{(5)}(0) = 24$

$\vdots \qquad \vdots$

Maclaurin Series

$$\cancel{f(0)}^{0} + \cancel{f'(0)}^{+1} x + \frac{\cancel{f''}^{-1}(0)}{2!}x^2 + \frac{\cancel{f'''}^{2}(0)}{3!}x^3 + \frac{\cancel{f^{(4)}}^{-6}(0)}{4!}x^4 + \frac{\cancel{f^{(5)}}^{24 = 4!}(0)}{5!}x^5 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{2x^3}{6} - \frac{6}{24}x^4 + \frac{4!}{5!}x^5 - \cdots$$

$$= \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots}$$

Matches

(b) Demonstrate a second, **different** method to compute the MacLaurin Series for $\boldsymbol{f(x) = \ln(1+x)}$. Justify. Do not just write down a formula.

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$

Test $x=0$

$$\underbrace{\ln(1+0)}_{0} = 0 - 0 + 0 - 0 \cdots + C \Rightarrow C = 0$$

Finally, $\ln(1+x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}}$ Matches

9. [10 Points]

(a) Write the **first 6 non-zero terms** of the MacLaurin Series for $\boldsymbol{f(x) = \sin(x^3) + \cos(x^3)}$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin(x^3) = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \cdots \qquad \cos(x^3) = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \cdots$$

$$= x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots \qquad = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \cdots$$

Combine in order

$$\sin(x^3) + \cos(x^3) = \boxed{1 + x^3 \boxed{- \frac{x^6}{2!}} \boxed{- \frac{x^9}{3!}} + \frac{x^{12}}{4!} + \frac{x^{15}}{5!} - \cdots}$$

$$\boxed{0}x^7 + 0x^8$$

(b) Use this series to now determine the **sixth, seventh, eighth** and **ninth** derivatives of $\boldsymbol{f(x) = \sin(x^3) + \cos(x^3)}$ evaluated at $x = 0$. Do Not Simplify your answers.

MacLaurin Series

$$f(0) + f'(0)x + \cdots + \boxed{\frac{f^{(6)}(0)}{6!}}x^6 + \boxed{\frac{f^{(7)}(0)}{7!}}x^7 + \boxed{\frac{f^{(8)}(0)}{8!}}x^8 + \boxed{\frac{f^{(9)}(0)}{9!}}x^9 + \cdots$$

$$\frac{f^{(6)}(0)}{6!} = \frac{-1}{2!} \Rightarrow f^{(6)}(0) = \boxed{\frac{-6!}{2!}} \qquad \frac{f^{(9)}(0)}{9!} = -\frac{1}{3!} \Rightarrow f^{(9)}(0) = \boxed{\frac{-9!}{3!}}$$

$$\frac{f^{(7)}(0)}{7!} = 0 \Rightarrow f^{(7)}(0) = \boxed{0}$$

$$\frac{f^{(8)}(0)}{8!} = 0 \Rightarrow f^{(8)}(0) = \boxed{0}$$

$$\frac{dx}{dt} = \frac{1}{1+t^2} - 1 \qquad \frac{dy}{dt} = \frac{2}{\sqrt{1+t^2}}$$

10. [18 Points]

(a) Consider the Parametric Curve represented by $x = (\arctan t) - t$ and $y = 2\sinh^{-1} t$.

$$\text{Recall} \quad \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$

COMPUTE the **arclength** of this parametric curve for $0 \le t \le \sqrt{3}$.

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} - 1\right)^2 + \underbrace{\left(\frac{2}{\sqrt{1+t^2}}\right)^2}_{\frac{4}{1+t^2}}}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2}\right)^2 - \underbrace{\frac{2}{1+t^2}}\ +1 + \underbrace{\frac{4}{1+t^2}}_{}}\, dt = \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2}\right)^2 + \frac{2}{1+t^2} + 1}\, dt$$

Combine

$$= \int_0^{\sqrt{3}} \cancel{\sqrt{\left(\frac{1}{1+t^2} + 1\right)^2}}\, dt = \int_0^{\sqrt{3}} \frac{1}{1+t^2} + 1\, dt$$

$$= \arctan t + t \Big|_0^{\sqrt{3}} = \underbrace{\arctan\sqrt{3}}_{\pi/3} + \sqrt{3} - (\underbrace{\arctan 0}_{0} + 0)$$

$$= \boxed{\frac{\pi}{3} + \sqrt{3}}$$

$\frac{dx}{dt} = 3\cos^2 t(-\sin t)$ $\frac{dy}{dt} = 3\sin^2 t \cos t$

10. (Continued)

(b) Consider a different Parametric Curve represented by $x = \cos^3 t$ and $y = \sin^3 t$.

COMPUTE the **surface area** obtained by rotating this curve about the y-axis for $0 \le t \le \frac{\pi}{2}$.

$$S.A = 2\pi \int_0^{\pi/2} x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= 2\pi \int_0^{\pi/2} \cos^3 t \sqrt{\underbrace{(-3\cos^2 t \sin t)^2}_{9\cos^4 t \sin^2 t} + \underbrace{(3\sin^2 t \cos t)^2}_{9\sin^4 t \cos^2 t}}\, dt$$

$$= 2\pi \int_0^{\pi/2} \cos^3 t \underbrace{\sqrt{9\cos^2 t \sin^2 t \underbrace{(\cos^2 t + \sin^2 t)}_{1}}}_{3\cos t \sin t}\, dt$$

$$= 6\pi \int_0^{\pi/2} \cos^4 t \sin t\, dt$$

$$= 6\pi \left(-\frac{\cos^5 t}{5}\right)\Big|_0^{\pi/2}$$

$$= -\frac{6\pi}{5}\left(\underbrace{\cos^5(\pi/2)}_{0} - \underbrace{\cos^5 0}_{1}\right)$$

$$= -\left(-\frac{6\pi}{5}\right) = \boxed{\frac{6\pi}{5}}$$

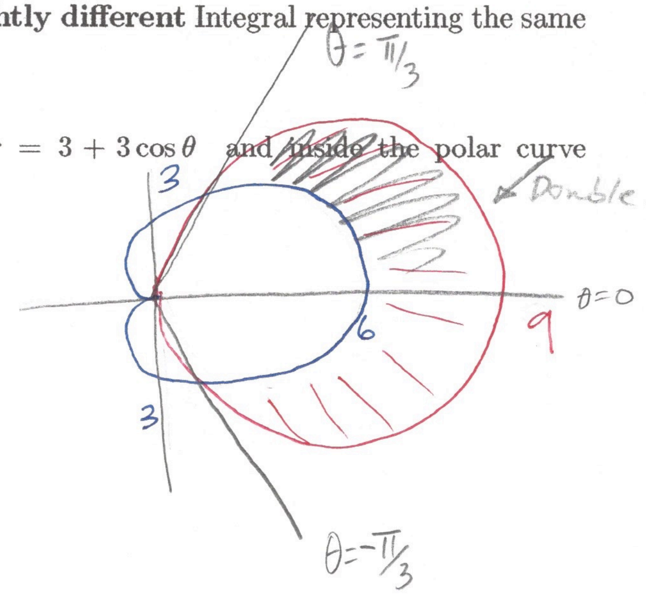
11. [20 Points] For **each** of the following problems, do the following **THREE** things:

1. Sketch the Polar curve(s) and shade the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **area** bounded outside the polar curve $r = 3 + 3\cos\theta$ and inside the polar curve $r = 9\cos\theta$.



$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, d\theta$$

$$= \boxed{\frac{1}{2}\int_{-\pi/3}^{\pi/3} (9\cos\theta)^2 - (3+3\cos\theta)^2 \, d\theta}$$

OR

$$= \boxed{2\left[\frac{1}{2}\int_{0}^{\pi/3} (9\cos\theta)^2 - (3+3\cos\theta)^2 \, d\theta\right]}$$

Double by Symmetry

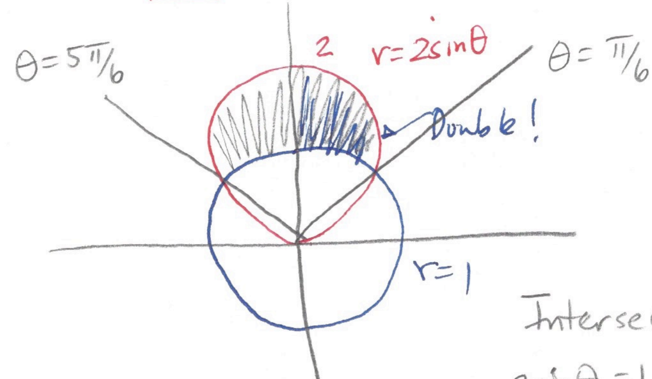
Intersect? $3 + 3\cos\theta = 9\cos\theta$

$6\cos\theta = 3$

$\cos\theta = 1/2$

$\theta = \pm \pi/3$

(b) The **area** bounded outside the polar curve $r = 1$ and inside the polar curve $r = 2\sin\theta$.



$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, d\theta$$

$$= \boxed{\frac{1}{2}\int_{\pi/6}^{5\pi/6} (2\sin\theta)^2 - (1)^2 \, d\theta}$$

OR

$$= \boxed{2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} (2\sin\theta)^2 - 1 \, d\theta\right]}$$

Double by Symmetry

Intersect?

$2\sin\theta = 1$

$\sin\theta = 1/2$

$\theta = \pi/6, 5\pi/6$

11. (Continued) For **each** of the following problems, do the following **THREE** things:

1. Sketch the Polar curve(s) and shade the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

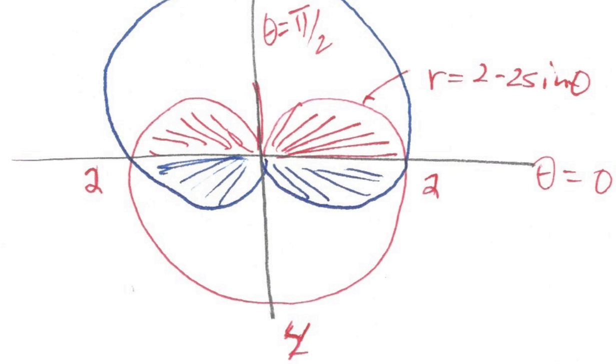
3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(c) The **area** that lies inside both of the curves $r = 2 + 2\sin\theta$ and $r = 2 - 2\sin\theta$.

$$A = 4\left[\frac{1}{2}\int_0^{\pi/2} (\text{Polar Radius})^2\, d\theta\right]$$

$$= \boxed{4\left[\frac{1}{2}\int_0^{\pi/2} (2-2\sin\theta)^2\, d\theta\right]}$$

Symmetry

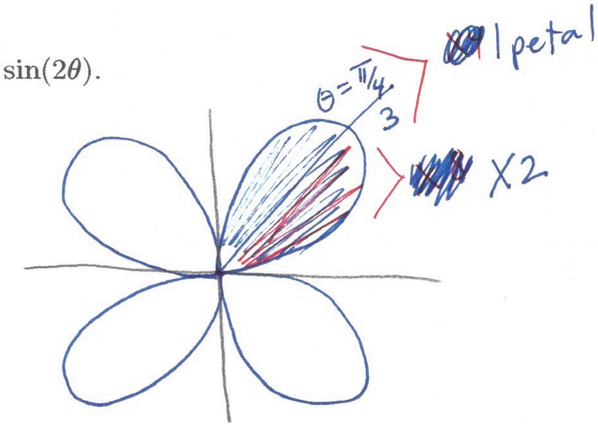


OR

$$= \boxed{2\left[\frac{1}{2}\int_0^{\pi} (2-2\sin\theta)^2\, d\theta\right]}$$

Symmetry

(d) The **area** bounded inside one petal of the curve $r = 3\sin(2\theta)$.



$$A = \frac{1}{2}\int_0^{\pi/2} (\text{Polar Radius})^2\, d\theta$$

$$= \boxed{\frac{1}{2}\int_0^{\pi/2} [3\sin(2\theta)]^2\, d\theta}$$

OR

$$= \boxed{2\left[\frac{1}{2}\int_0^{\pi/4} (3\sin(2\theta))^2\, d\theta\right]}$$

Double by Symmetry