Integral Test

Consider a series of the form $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$ where the terms $a_n = f(n)$ and the related function $f(x)$ is continuous, positive, and decreasing on $[1, \infty)$.

1. If the $\int_1^{\infty} f(x)\, dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If the $\int_1^{\infty} f(x)\, dx$ diverges, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

USED: For positive-termed series. It is only as helpful as it is possible to compute the improper integral $\int_1^{\infty} f(x)\, dx$.

NOTE: Do not worry if the related function is only *eventually* decreasing. For instance, if $f(x)$ is decreasing for $x > 3$, the theorem will still work, because the first few terms that do not fit the statement are still finite. It is always the tail end terms of the given series that we are concerned with.

NOTE: The Integral Test says that the Series and the Improper Integral either both converge or both diverge. They both **share the same convergence behavior**.

NOTE: Usually used when no other, simpler, convergence tests applies. This test can be helpful for $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ or $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$.

WARNING: The series sum is **not** equal to the value from the improper integral. They are merely comparable in size.

APPROACH:

- Given the positive termed series, pick off the related function $f(x)$. You **must** check that the three conditions are satisfied by f before using this convergence test.
- To prove that the related function is decreasing, you can compute the derivative $f'(x)$ and prove that it is negative. You can also make a general, detailed size argument. It is not enough justification to simply list the first few terms of the sequence a_n.
- Compute the improper integral $\int_1^{\infty} f(x)\, dx$. Be sure to handle the *improperness* of the integral, and clearly write out the limiting value. State the final value of the improper integral. Be clear what the numerical answer is if it is finite or equals $\pm\infty$.
- Make **two** clear conclusions. First state whether the improper integral converges or diverges. Second state whether the original series converges or diverges. (This is really the ultimate answer of interest, but it is determined by the first statement of convergence or divergence of the integral.)

EXAMPLES: Determine and state whether each of the following series **converges** or **diverges**. Name any convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

The related function $f(x) = \dfrac{x}{x^4+1}$ is continuous for all real x, postive for $x > 0$, and decreasing because $f'(x) = \dfrac{-3x^4+1}{(x^4+1)^2} < 0$ certainly for $x \geq 1$, technically $x \geq \sqrt[4]{\frac{1}{3}}$.

Compute $\displaystyle\int_1^{\infty} \frac{x}{x^4+1}\, dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\, dx = \lim_{t\to\infty} \int_1^t \frac{x}{\left(x^2\right)^2+1}\, dx = \lim_{t\to\infty} \frac{1}{2} \int_1^{t^2} \frac{1}{u^2+1}\, du$

$\displaystyle = \lim_{t\to\infty} \frac{1}{2} \arctan u \Big|_1^{t^2} = \lim_{t\to\infty} \frac{1}{2} \left(\arctan\left(t^2\right) - \arctan 1\right) = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$

Here $\boxed{\begin{array}{l} u = x^2 \\ du = 2x dx \\ \frac{1}{2} du = dx \end{array}}$ and $\boxed{\begin{array}{l} x = 1 \Rightarrow u = 1 \\ x = t \Rightarrow u = t^2 \end{array}}$

The improper integral converges. Finally, the Original Series also $\boxed{\text{converges}}$ by the Integral Test.

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

The related function $f(x) = \dfrac{1}{x \ln x}$ is continuous for $x > 1$, postive for $x > 1$, and decreasing for $x \geq 1$ because $f'(x) = \dfrac{-(1+\ln x)}{(x \ln x)^2} < 0$ when $1 + \ln x > 0$ or $\ln x > -1$ or when $x > e^{-1}$.

Compute $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\, du = \lim_{t\to\infty} \ln|u| \Big|_{\ln 2}^{\ln t}$

$\displaystyle = \lim_{t\to\infty} \ln|\ln t| - \ln|\ln 2| = \infty$ Here $\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array}}$ and $\boxed{\begin{array}{l} x = 2 \Rightarrow u = \ln 2 \\ x = t \Rightarrow u = \ln t \end{array}}$

The improper integral diverges. Finally, the Original Series also $\boxed{\text{diverges}}$ by the Integral Test.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$

The related function $f(x) = \dfrac{e^{\frac{1}{x}}}{x^2}$ is continuous for $x > 0$, postive for $x > 0$, and decreasing for $x \geq -\frac{1}{2}$ (certainly $x \geq 1$) because $f'(x) = \dfrac{-e^{\frac{1}{x}}(1+2x)}{x^4} < 0$ then.

Compute $\displaystyle\int_1^{\infty} \frac{e^{\frac{1}{x}}}{x^2}\, dx = \lim_{t\to\infty} \int_1^t \frac{e^{\frac{1}{x}}}{x^2}\, dx = \lim_{t\to\infty} -e^{\frac{1}{x}} \Big|_1^t = \lim_{t\to\infty} -e^{\frac{1}{t}} + e = e - 1$

The improper integral converges. Finally, the Original Series also $\boxed{\text{converges}}$ by the Integral Test.