Geometric Series Test

Consider a series of the form $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$. This geometric series

$$\begin{cases} \text{converges if } |r| < 1, \quad \text{with SUM} = \dfrac{a}{1-r} \\ \\ \text{diverges if } |r| \geq 1 \end{cases}$$

USED: For series where each successive term is found by multiplying the previous term by a common ratio r. These series contain terms that look "exponential-ish" where there is a fixed base raised to changing or variable powers.

NOTE: Do not worry if the given power is not exactly of the form $n-1$. You do **not** need to muscle your series terms to match the $n-1$ form. In fact that usually leads to more mistakes.

APPROACH:

- Write out at least the first three terms for a geometric series. The first term is a, and the multiplying factor to compute each successive term is the common ratio r. Technically you only need the first two terms to find a and r, but it is strongly recommended to write a third term so you can confirm that you have indeed chosen the correct r.
- Determine if $|r|$ is less than 1 or greater than or equal to 1. The test says you must check the **absolute value** or r. Even if r is positive, it is recommended to write $|r|$ because that is the condition in the convergence test. Finally compare $|r|$ to 1, and make a clear declaration of convergence or divergence.
- If $|r| \geq 1$ so that the series diverges, then you are done. If $|r| < 1$, so that the series converges, then you can compute the actual sum of the full original geometric series. Here the SUM$=\dfrac{a}{1-r}$. Please simplify.

EXAMPLES: Determine and state whether each of the following series **converges** or **diverges**. Name any convergence test(s) that you use, and justify all of your work. If the geometric series converges, compute the sum.

1. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^{n+2}}{2^{3n-1}} = -\frac{3^3}{2^2} + \frac{3^4}{2^5} - \frac{3^5}{2^8} + \ldots$

Here we have a geometric series with $a = -\dfrac{27}{4}$ and $r = -\dfrac{3}{2^3} = -\dfrac{3}{8}$. It $\boxed{\text{converges}}$ as a geometric series (or by GST) since $|r| = \left|-\dfrac{3}{8}\right| = \dfrac{3}{8} < 1$.

As a result, the sum is given by SUM$=\dfrac{a}{1-r} = \dfrac{-\frac{27}{4}}{1-\left(-\frac{3}{8}\right)} = \dfrac{-\frac{27}{4}}{\frac{11}{8}} = -\dfrac{27}{4}\cdot\dfrac{8}{11} = \boxed{-\dfrac{54}{11}}$

2. $\sum_{n=1}^{\infty}(-1)^n \frac{4^{3n-1}}{7^{n+3}} = -\frac{4^2}{7^4} + \frac{4^5}{7^5} - \frac{4^8}{7^6} + \dots$

Here we have a geometric series with $r = -\frac{4^3}{7} = -\frac{64}{7}$. It $\boxed{\text{diverges}}$ as a geometric series (or by GST) since $|r| = \left|-\frac{64}{7}\right| = \frac{64}{7} > 1$.

(Note you don't need a here because the series diverges and you will NOT be computing the sum.)

3. $\sum_{n=1}^{\infty}(-1)^n \frac{4^{2n+1}}{3^{4n-1}} = -\frac{4^3}{3^3} + \frac{4^5}{3^7} - \frac{4^7}{3^{11}} + \dots$

Here we have a geometric series with $a = -\frac{64}{27}$ and $r = -\frac{4^2}{3^4} = -\frac{16}{81}$. It $\boxed{\text{converges}}$ as a geometric series (or by GST) since $|r| = \left|-\frac{16}{81}\right| = \frac{16}{81} < 1$.

As a result, the sum is given by SUM$=\frac{a}{1-r} = \frac{-\frac{64}{27}}{1-\left(-\frac{16}{81}\right)} = \frac{-\frac{64}{27}}{\frac{97}{81}} = -\frac{64}{27}\cdot\frac{81}{97} = \boxed{-\frac{192}{97}}$