Alternating Series Test

Consider a series $\sum_{n=1}^{\infty}(-1)^{n+1}\, b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \dots$ with terms alternating in sign. Then if the following three conditions are all satisfied:

$$\begin{cases} 1.\ b_n > 0 \\ 2.\ \lim_{n\to\infty} b_n = 0 \\ 3.\ b_{n+1} \le b_n \end{cases} \quad \text{then the alternating series } \sum_{n=1}^{\infty}(-1)^{n+1}\, b_n \text{ converges.}$$

USED: On Alternating Series where the terms are shrinking appropriately and when the Ratio Test is not appplicable or when analyzing the Absolute Series and using the Absolute Convergence Test is not helpful. Helpful when showing a series is Conditionally Convergent.

NOTE: Condition 3 is verified by making a convincing size argument for general n, or by showing the related function is decreasing. (how?)

WARNING: If condition 2 is not satisfied, then the series would have diverged by the n^{th} Term Divergence Test. Here if $\lim_{n\to\infty} b_n \neq 0$, then that implies that the original terms $\lim_{n\to\infty}(-1)^{n+1}b_n \neq 0$.

WARNING: You **never** get a declaration of divergence out of the AST. The failing of condition 2 might lead you to a declaration of divegrence from the n^{th} Term Divergence Test, but AST itself does not give divergence.

APPROACH:

- Given the original series, pick off the positive portion of given terms, b_n. Simply confirm that you have done that correctly. Check that $b_n > 0$ for $n \ge 1$.
- Prove that the positive b_n term pieces are approaching 0 as $n \to \infty$. No guess here. Justify with L'H Rule, using the related function, if needed. **IF the b_n terms do not approach 0, STOP running this test, and go make a conclusion from the n^{th} term divergence test as described above.**
- Prove that the terms are decreasing by showing an obvious size argument for general n, or prove that the derivative of the related function is negative.
- Make a conclusion about the original series.

EXAMPLES: Determine and state whether each of the following series **converges** or **diverges**. Name any convergence test(s) that you use, and justify all of your work.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{n^2+1}$

(1) Isolate $b_n = \dfrac{n}{n^2+1} > 0$ for $n \geq 1$

(2) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} \cdot \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = 0$

(3) $b_{n+1} < b_n$ since we can show the derivative of the related function is negative, hence the terms are decreasing. Consider $f(x) = \dfrac{x}{x^2+1}$ with $f'(x) = \dfrac{-x^2+1}{(x^2+1)^2} < 0$ for $x > 1$

Therefore, the original series $\boxed{\text{converges}}$ by the Alternating Series Test.

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^3+1}$ Attempt AST here.

(1) Pick off $b_n = \dfrac{n^3}{n^3+1} > 0$ for $n \geq 1$.

(2) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^3}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^3}} = 1 \neq 0$ **STOP AST NOW!**

This implies that the original terms have $\displaystyle\lim_{n\to\infty} (-1)^n \frac{n^3}{n^3+1} \neq 0$ (in fact it's DNE). Finally the O.S. $\boxed{\text{diverges}}$ by the n^{th} Term Divergence Test.

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+3}$

(1) Examine $b_n = \dfrac{\sqrt{n}}{n+3} > 0$ for $n \geq 1$

(2) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+3} \cdot \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1+\frac{3}{n}} = 0$

(3) $b_{n+1} < b_n$ since we can show the derivative of the related function is negative, hence the terms are decreasing. Consider $f(x) = \dfrac{\sqrt{x}}{x+3}$ with $f'(x) = \dfrac{3-x}{2\sqrt{x}(x+3)^2} < 0$ for $x > 3$

Therefore, O.S. $\boxed{\text{Conv.}}$ by AST.