Math 121 Exam 3
April 26, 2024

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{3\ln 3}$, or $\arctan\sqrt{3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [16 Points] Find the **Interval** and **Radius** of Convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x+5)^n}{(3n+5)^2 \cdot 4^n}$

Analyze carefully and with full justification.

2. [22 Points] Find the MacLaurin Series for each of the functions. Also **STATE** the Radius of Convergence for each series. Answers should be in sigma notation $\displaystyle\sum_{n=0}^{\infty}$. Simplify.

(a) $\ln\left(1+9x^2\right)$ (b) x^3e^{-4x} (c) $\dfrac{d}{dx}\left(8x^4\sin(8x)\right)$ (d) $\displaystyle\int \frac{x^2}{8+x^3}\,dx$

3. [12 Points] Use MacLaurin Series to Estimate $\cos\left(\dfrac{1}{2}\right)$ with error less than $\dfrac{1}{10000}$.

Simplify.

Tips: $(16)\cdot(24) = 384$ and $6! = 720$ and $(720)\cdot(64) = 46,080$ and $(48)\cdot 8 = 384$

4. [26 Points] Find the **sum** for each of the following convergent series. Simplify, if possible.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\,\pi^{2n}}{3^{2n}\,(2n+1)!}$ (b) $-1+\dfrac{1}{2}-\dfrac{1}{3}+\dfrac{1}{4}-\dfrac{1}{5}+\dfrac{1}{6}-\dots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\,\pi^{2n+1}}{4^n\,(2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(\ln 9)^n}{2^n \cdot n!}$ (e) $4+4-\dfrac{4}{3}+\dfrac{4}{5}-\dfrac{4}{7}+\dfrac{4}{9}-\dots$ (f) $-\dfrac{\pi^2}{2!}+\dfrac{\pi^4}{4!}-\dfrac{\pi^6}{6!}+\dfrac{\pi^8}{8!}-\dots$

5. [12 Points] Use Series to compute the following Limit $\lim_{x \to 0} \dfrac{1 - \cos(2x)}{e^{-x} - 1 + x}$

6. [12 Points] Find the MacLaurin Series Representation for $\arctan\left(x^4\right)$.
You **MUST** use Integration of Series, and the following Hint formula.

Hint: $\boxed{\arctan\left(x^4\right) = \int \frac{4x^3}{1 + x^8}\, dx = \int 4x^3 \left(\frac{1}{1 + x^8}\right)\, dx}$ (P.S.) Yes, solve for C

OPTIONAL: Feel free to check your answer using a substitution.

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Create a Power Series with Interval of Convergence $I = \left[-\frac{3}{2}, \frac{4}{5}\right]$ Continue on to justify that your series satisifies this challenge.