Math 121 Exam 3
December 4, 2024

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{3\ln 3}$, or $\arctan\sqrt{3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [20 Points] Interval and Radius of Convergence. Analyze carefully, with full justification.

(a) Find the **Interval** and **Radius** of Convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3x+5)^n}{(3n+5)^2 \cdot 4^n}$

(b) Design a Power Series which is convergent only at $x = 6$. Once you create your series, then proceed to justify that the Interval of Convergence is indeed $I = \{6\}$.

2. [22 Points] Find the MacLaurin Series for each of the functions. Also **STATE** the Radius of Convergence for each series. Answers should be in sigma notation $\displaystyle\sum_{n=0}^{\infty}$. Simplify.

(a) $\ln(1 + 9x^2)$ (b) $x^3 e^{-4x}$ (c) $\dfrac{d}{dx}\left(8x^4 \sin(8x)\right)$ (d) $\displaystyle\int \frac{x^2}{8 + x^3}\,dx$

3. [10 Points] Use Series to Estimate $\dfrac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$ with error less than $\dfrac{1}{100}$.

Simplify. Tip: $(16) \cdot (24) = 384$

4. [26 Points] Find the **sum** for each of the following convergent series. Simplify, if possible.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{3^{2n}\ (2n+1)!}$ (b) $\displaystyle -1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\dots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n+1}}{4^n\ (2n)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ (\ln 9)^n}{2^n \cdot n!}$ (e) $\displaystyle 4+4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\dots$ (f) $\displaystyle -\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\frac{\pi^8}{8!}-\dots$

5. [10 Points] Use Series to compute the following Limit $\displaystyle\lim_{x\to 0} \frac{1-\cos(2x)}{e^{-x}-1+x}$

6. [12 Points] Find the MacLaurin Series Representation for $\arctan\left(x^4\right)$.
You **MUST** use Integration of Series, and the following Hint formula.

$$\text{Hint: } \boxed{\arctan\left(x^4\right) = \int \overset{\text{P.S.}}{\frac{4x^3}{\cancel{1}+x^8}}\, dx = \int 4x^3 \left(\frac{1}{1+x^8}\right)\, dx} \quad \text{Yes, solve for } C$$

OPTIONAL: Feel free to check your answer using a substitution.