Math 12 Final Exam May 11, 2011

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 0} \frac{1-\cosh(2x)}{x+\ln(1-x)}$ (b) $\displaystyle\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{1}{(x^2+4)^{\frac{5}{2}}}dx$ (b) $\displaystyle\int x \arcsin x \, dx$

(c) $\displaystyle\int \frac{x^4+2x^3+7x^2+8x+7}{x^3+x^2+4x+4}\,dx = \int \frac{x^4+2x^3+7x^2+8x+7}{(x+1)(x^2+4)}\,dx$

3. [20 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_7^\infty \frac{1}{x^2-6x+25}\,dx$ (b) $\displaystyle\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$

4. [10 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4^{n+1}}{3^{2n-1}}$ (b) $1 - \ln 7 + \frac{(\ln 7)^2}{2!} - \frac{(\ln 7)^3}{3!} + \frac{(\ln 7)^4}{4!} - \frac{(\ln 7)^5}{5!} + \ldots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n)!}$

5. [20 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, n}{3n^2+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+4^n}{5^{n+1}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n + n^2\sqrt{n}}{n^7+1}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3n)!}{n^n\, 2^n\, (n!)^2\, \ln n}$

6. [10 Points] Find the **Interval** and **Radius** of Convergence for the power series $\sum_{n=0}^{\infty} \frac{(-1)^n (2x-3)^n}{n\ 6^{n+1}}$.

Analyze carefully and with full justification.

7. [5 Points] Write the MacLaurin Series for $\boldsymbol{f(x) = e^{-x^2}}$. Use this series to determine the **fourth** and **fifth** derivatives of $f(x) = e^{-x^2}$ at $x = 0$.

8. [10 Points] Please analyze with detail and justify carefully.

(a) Find the **MacLaurin series** representation for $\boldsymbol{f(x) = x \arctan x}$. Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(b) Use the MacLaurin series representation for $\boldsymbol{f(x) = x \arctan x}$ from Part(a) to

$$\textbf{Estimate} \quad \int_0^{\frac{1}{2}} x \arctan x \, dx \quad \text{with error less than } \frac{1}{100}.$$

Justify in words that your error is indeed less than $\frac{1}{100}$.

9. [15 Points] Consider the region bounded by $\boldsymbol{y = \cos x}$, $\boldsymbol{y = \sin x}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = \frac{\pi}{4}}$. Rotate the region about the $\boxed{y\text{–axis}}$. **Compute** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [15 Points] Consider the Parametric Curve represented by $\boldsymbol{x = e^t - t}$ and $\boldsymbol{y = 4e^{t/2}}$.

(a) Compute the **arclength** of this parametric curve for $\boldsymbol{0 \le t \le 1}$.

(b) Set-up, **BUT DO NOT EVALUATE**, the definite integral representing the **surface area** obtained by rotating this curve about the ***y*-axis**, for $\boldsymbol{0 \le t \le 1}$.

11. [15 Points] Compute the **area** bounded inside the polar curve $\boldsymbol{r = 2 + 2\cos\theta}$ and outside the polar curve $\boldsymbol{r = 3}$. **Sketch** the Polar curves.

12. [10 Points] Find the general solution for each of the following **differential equations**.

(a) $\boldsymbol{\frac{dy}{dx} = (\ln x)\sqrt{1 - y^2}}$

(b) $\boldsymbol{x \frac{dy}{dx} - y = x^2 e^x}$