**Amherst College, DEPARTMENT OF MATHEMATICS**

**Math 12 (Section 01), Final Exam, May 13, 2010**

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

**1.** [10 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to 3} \frac{\arctan(x-3)}{x^2-9}$ (b) $\lim_{x\to\infty} (e^x+1)^{\frac{1}{x}}$

**2.** [15 Points] Evaluate each of the following **integrals**.

(a) $\int_0^{\ln 7} \frac{\sinh x}{\cosh x} dx$ (b) $\int x \arcsin x \, dx$

**3.** [20 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\int_7^{\infty} \frac{1}{x^2-8x+19}\, dx$ (b) $\int_0^4 \frac{3x+2}{x^2-x-12}\, dx$

**4.** [10 Points] Find the **sum** of each of the following series (which do converge):

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{9^{n-1}}$ (b) $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n+1)!}$

**5.** [20 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

(b) $\sum_{n=1}^{\infty} \frac{e^n+9}{n+1}$ (d) $\sum_{n=1}^{\infty} (-1)^n \frac{n^3+8}{n^7-9}$

**6.** [10 Points] Find the **Interval** and **Radius** of Convergence for the following power series $\sum_{n=0}^{\infty} \frac{(-1)^n(3x-1)^n}{n^2\cdot 2^n}$. Analyze carefully and with full justification.

**7.** [5 Points] Consider the function $f(x)$ that satisfies the following: $f(3) = 2, f'(3) = -3, f''(3) = \frac{6}{7}$, and $f'''(3) = -1$. Find the **Taylor polynomial of degree 3** for $\mathbf{f(x)}$ centered at $\mathbf{a = 3}$.

**8.** [10 Points] **MacLaurin Series:** Please analyze with detail and justify carefully.

(a) Find the **MacLaurin series** representation for $\boldsymbol{f(x) = x \ln(1+x)}$.

Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(b) Use the MacLaurin Power Series representation for $\boldsymbol{f(x) = x \ln(1+x)}$ from Part(a) to

$$\textbf{Estimate} \quad \int_0^1 x \ln(1+x)\, dx \quad \text{with error less than } \frac{1}{10}.$$

**9.** [15 Points] Volumes of Revolution

(a) Consider the region bounded by $\boldsymbol{y = e^x}$, $\boldsymbol{y = \ln x}$, $\boldsymbol{x = 1}$, and $\boldsymbol{x = 2}$. Rotate the region about the $\boxed{\boldsymbol{y}\text{–axis}}$. **Compute** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(b) Consider the same region bounded by $\boldsymbol{y = e^x}$, $\boldsymbol{y = \ln x}$, $\boldsymbol{x = 1}$, and $\boldsymbol{x = 2}$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = -3}}$. Set up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method.

**10.** [15 Points] Consider the Parametric Curve represented by $\boldsymbol{x = e^t - t}$ and $\boldsymbol{y = 4e^{t/2}}$.

(a) Find the **arclength** of this parametric curve for $0 \le t \le 1$.

(b) Set up, **BUT DO NOT EVALUATE!!**, the definite integral representing the **surface area** of the solid obtained by rotating this curve about the $x$-**axis**, for $0 \le t \le 1$.

**11.** [15 Points] For each part, **sketch** the Polar curve(s), and answer the related questions:

(a) **Compute** the **area** enclosed by the cardioid $\boldsymbol{r = 1 + \sin\theta}$.

(b) Set up, **BUT DO NOT EVALUATE!!**, the definite integral representing the **area** bounded inside $\boldsymbol{r = 2 + 2\cos\theta}$ and outside $\boldsymbol{r = 3}$.

**12.** [10 Points] Find the general solution for each of the following **differential equations**.

(a) $\dfrac{dy}{dx} = (1 + y^2)e^x$

(b) $\dfrac{dy}{dx} + 2xy = e^{-x^2}$