Math 121 Final May 12, 2025

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\arctan(\sqrt{3})$, should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [16 Points] (a) Use Series to show that $\lim_{x\to 0} \frac{\cos(3x) - 1 - \arctan(2x) + 2x}{e^{-4x} - 1 + 4x} = \boxed{-\frac{9}{16}}$

(b) Compute $\lim_{x\to 0} \frac{\cos(3x) - 1 - \arctan(2x) + 2x}{e^{-4x} - 1 + 4x}$ again using L'Hôpital's Rule.

2. [18 Points] Compute the following integral. Simplify.

(a) $\int \frac{1}{(x^2+4)^2}\, dx$ (b) $\int x^4 \arcsin x \, dx$

3. [24 Points] For each of the following **Improper** integrals, determine whether it Converges or Diverges. If it converges, find its value. Simplify.

(a) $\int_0^e x^3 \ln x \, dx$ (b) $\int_{-\infty}^1 \frac{1}{x^2 - 6x + 13}\, dx$ (c) $\int_{-7}^0 \frac{x+15}{x^2+6x-7}\, dx$

4. [24 Points] Find the **Sum** of each of the following series (which do converge). Simplify.

(a) $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n\, (\ln 9)^n}{n!}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\, \pi^{2n}}{9^n\, (2n+1)!}$

(d) $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots$ (e) $1 + 1 + 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \ldots$ (f) Show $\sum_{n=0}^{\infty} \frac{(-4)^n - 2}{5^n} = \boxed{-\frac{35}{18}}$

5. [26 Points] (a) Find the Interval/Radius of Convergence for $\sum_{n=1}^{\infty} \frac{(-1)^n\, (5x+7)^n}{(5n+7)^2\, 8^n}$

(b) Show that the Series for $\cos x$ has Infinite Radius of Convergence, $R = \infty$.

(c) Create a Power Series centered at $a = 5$ which has a Radius of Convergence, $R = 0$. Once you create your series, then proceed to justify that the Interval of Convergence is indeed $I = \{5\}$.

6. [30 Points] Determine whether each series **Converges** or **Diverges**. Name any Convergence Test(s) you use; justify all of your work.

(a) Create a Series that **Diverges by the n^{th} Term Divergence Test and** requires L'Hôpital's Rule in the nTDT Limit. Continue on to justify that the series is Divergent by the nTDT.

(b) Create an Alternating Series that **Converges by the Absolute Convergence Test**. You cannot choose a series of *just* the form $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$. Continue on to justify that the series is Convergent by the ACT.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Use the Integral Test (d) $\displaystyle\sum_{n=1}^{\infty} 7$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^7} + \frac{(-5)^n}{7^{2n}}$

7. [18 Points] Determine whether the series is **Absolutely Convergent** or **Conditionally Convergent**. Name any Convergence Test(s) you use, justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^5+7}{n^7+5}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{7n+5}$

8. [8 Points] Use Series to Estimate $\sin\left(\frac{1}{2}\right)$ with error less than $\frac{1}{2,000}$. Simplify.

Tip: $(120) \cdot (32) = 3840$

9. [10 Points] Answer in Sigma $\displaystyle\sum_{n=0}^{\infty}$ notation. **State** the Radius of Convergence

Find the MacLaurin Series for $\ln(4+x^2)$. Use the Hint: $\ln(4+x^2) = \displaystyle\int \frac{2x}{4+x^2}\, dx = \ldots$.

Yes, solve for C

10. [8 Points] **COMPUTE** the Area bounded inside the Cardioid $r = 1 - \cos\theta$. Sketch and shade the bounded region.

11. [18 Points] For **each** of the following problems, do the following **THREE** things:

1. **Sketch** the Polar curve(s) and **Shade** the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **Area** bounded Outside the polar curve $r = 2 + 2\cos\theta$ and Inside $r = 6\cos\theta$.

(b) The **Area** bounded Inside **Both** of the curves $r = 2 + 2\sin\theta$ and $r = 2 - 2\sin\theta$.

(c) The **Area** bounded Outside the polar curve $r = 1 - \sin\theta$ and Inside $r = -3\sin\theta$.