Math 121 Final May 17, 2024

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\arctan(\sqrt{3})$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [12 Points] (a) Use Series to show that $\displaystyle\lim_{x\to 0} \frac{x^2+4x-\arctan(4x)}{1-3x-e^{-3x}} = \boxed{-\frac{2}{9}}$

(b) Compute $\displaystyle\lim_{x\to 0} \frac{x^2+4x-\arctan(4x)}{1-3x-e^{-3x}}$ again using L'Hôpital's Rule.

2. [18 Points] Compute the following integrals. Simplify.

(a) $\displaystyle\int \frac{x^3}{(x^2+4)^{\frac{7}{2}}}\,dx$ Use Trig Sub and Hint: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (b) $\displaystyle\int x\arcsin x\,dx$

3. [30 Points] For each of the following **Improper** integrals, determine whether it Converges or Diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_0^e \ln x\,dx$ (b) $\displaystyle\int_0^{e^4} \frac{4}{x\,[16+(\ln x)^2]}\,dx$ (c) $\displaystyle\int_{-4}^{-3} \frac{8-x}{x^2+2x-8}\,dx$ (d) $\displaystyle\int_5^{\infty} \frac{7}{x^2-4x+7}\,dx$

4. [24 Points] Find the **Sum** of each of the following series (which do converge). Simplify.

(a) $\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}+\ldots$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 9)^n}{3!\ 2^n\ n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ \pi^{2n}}{(36)^n\ (2n+1)!}$

(d) $-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\ldots$ (e) $1+1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\frac{\pi^8}{8!}-\ldots$ (f) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n-2}{5^n} = \boxed{-\frac{35}{18}}$

5. [22 Points] Find the **Interval** and **Radius** of Convergence for (a) and (b).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (4x+1)^n}{(4n+1)^2\ \ 7^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-7)^n}{n^n}$

(c) Design a Power Series which is Convergent **only** at $x=4$. Once you create your series, then proceed to justify that the Interval of Convergence is indeed $I=\{4\}$.

6. [25 Points] Determine whether each series **Converges** or **Diverges**. Name any convergence test(s) you use, justify all of your work.

(a) $\sum_{n=2}^{\infty} \left(1 - \frac{7}{n^4}\right)^{n^4}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+7}$ Use Absolute Conv.Test

(c) $\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$ Use Integral Test (d) $\sum_{n=2}^{\infty} \frac{n^4}{\ln n}$ (e) $\sum_{n=1}^{\infty} 7$ (f) $\sum_{n=1}^{\infty} \frac{1}{7^n} + \frac{4}{n^7}$

7. [18 Points] Determine whether each given series is **Absolutely Convergent** or **Conditionally Convergent**. Name any convergence test(s) you use, justify all of your work.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^4+7}{n^7+4}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+7}$

8. [7 Points] Use Series to Estimate $\int_0^1 x^3 \sin\left(x^2\right)\, dx$ with error less than $\frac{1}{10,000}$. Simplify.

Tips: $(120) \cdot (14) = 1680$ and $(60) \cdot (28) = 1680$ and $7! = 5040$ and $(5040) \cdot (18) = 90,720$

9. [20 Points] Answers in Sigma $\sum_{n=0}^{\infty}$ notation. **STATE** the Radius of Convergence for each.

(a) Demonstrate **TWO** different methods for deriving the MacLaurin Series for $\cos x$.

You may use the MacLaurin Series formula for $\sin x$ without extra justification.

(b) Find the MacLaurin Series Representation for $\ln(9 + x^2)$.

Use this formula: $\ln(9+x^2) = \int \frac{2x}{9+x^2}\, dx$. Yes, solve for C

(c) Show that the MacLaurin Series for $\frac{1}{2}\left(e^x - e^{-x}\right)$ equals $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ Justify.

10. [24 Points] For **each** of the following problems, do the following **THREE** things:

1. **Sketch** the Polar curve(s) and **Shade** the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **Area** bounded Outside the polar curve $r = 2 + 2\sin\theta$ and Inside $r = 6\sin\theta$.

(b) The **Area** bounded Inside **Both** of the curves $r = 2 + 2\cos\theta$ and $r = 2 - 2\cos\theta$.

(c) The **Area** bounded Inside **Both** of the polar curves $r = 2\cos\theta$ and $r = 2\sin\theta$.

(d) The **Area** bounded Outside $r = 1 - \cos\theta$ and Inside $r = -3\cos\theta$.