Math 121 Final May 16, 2023

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\arctan(\sqrt{3})$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] (a) Use Series to show that $\lim_{x\to 0} \dfrac{\ln(1+x) - \arctan x}{1 - 3x - e^{-3x}} = \boxed{\dfrac{1}{9}}$

(b) Compute $\lim_{x\to 0} \dfrac{\ln(1+x) - \arctan x}{1 - 3x - e^{-3x}}$ again using L'Hôpital's Rule.

2. [20 Points] Compute the following integrals. Simplify.

(a) $\displaystyle\int \frac{x^3}{(x^2+4)^{\frac{7}{2}}}\,dx$ Use Trig Sub and Hint: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (b) $\displaystyle\int x \arcsin x\,dx$

3. [32 Points] For each of the following **Improper** integrals, determine whether it Converges or Diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_0^e \ln x\,dx$ (b) $\displaystyle\int_0^{e^4} \frac{4}{x\left[16 + (\ln x)^2\right]}\,dx$ (c) $\displaystyle\int_{-4}^{-3} \frac{8-x}{x^2+2x-8}\,dx$ (d) $\displaystyle\int_5^{\infty} \frac{7}{x^2-4x+7}\,dx$

4. [24 Points] Find the **Sum** of each of the following series (which do converge). Simplify.

(a) $-\frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \ldots$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\,(\ln 9)^n}{3!\;2^n\;n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,\pi^{2n}}{(36)^n\,(2n+1)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,3^n}$ (e) $1 + 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots$ (f) Show that $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n - 2}{5^n} = \boxed{-\frac{35}{18}}$

5. [20 Points] Find the **Interval** and **Radius** of Convergence for each of the following.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(4x+1)^n}{(4n+1)^2\;7^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!\,(\ln n)\,(x-7)^n}{n^n}$

6. [25 Points] Determine whether each series **Converges** or **Diverges**. Name any convergence test(s) you use, justify all of your work.

(a) $\displaystyle\sum_{n=2}^{\infty} \left(1 - \frac{7}{n^4}\right)^{n^4}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+7}$ Use the Absolute Convergence Test

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$ Use the Integral Test (no pre-conditions check)

7. [22 Points] Determine whether each given series is **Absolutely Convergent** or **Conditionally Convergent**. Name any convergence test(s) you use, justify all of your work.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^4+7}{n^7+4}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+7}$

8. [18 Points] Answers in Sigma $\displaystyle\sum_{n=0}^{\infty}$ notation. **STATE** the Radius of Convergence for each.

(a) Let $f(x) = 3x^2 \cos(x^3)$. Find its Series in two ways.

(*) First, Replace $f(x) = 3x^2 \cos(x^3)$ directly with a Series.

(**) Second, Replace $\sin(x^3)$ with a Series and then Differentiate to find a series for $f(x)$ using the fact that $3x^2 \cos(x^3) = \dfrac{d}{dx} \sin\left(x^3\right)$.

Show that they are equal Series using the two different methods.

(b) Find the MacLaurin Series Representation for $\ln(9+x^2)$. Hint: $\ln(9+x^2) = \displaystyle\int \frac{2x}{9+x^2}\, dx$.

Yes, solve for C

(c) Show that the MacLaurin Series for $\dfrac{1}{2}\left(e^x - e^{-x}\right)$ equals $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ Justify.

9. [24 Points] For **each** of the following problems, do the following **THREE** things:

1. **Sketch** the Polar curve(s) and **Shade** the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **Area** bounded outside the polar curve $r = 2 + 2\sin\theta$ and inside $r = 6\sin\theta$.

(b) The **Area** that lies inside both of the curves $r = 2 + 2\cos\theta$ and $r = 2 - 2\cos\theta$.

(c) The **Area** bounded inside both of the polar curves $r = 2\cos\theta$ and $r = 2\sin\theta$.

(d) The **Area** bounded outside $r = 1 - \cos\theta$ and inside $r = -3\cos\theta$.