Math 121 Final Exam May 7, 2018

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [18 Points] Evaluate each of the following **limits**. Please justify your answer. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist. Simplify.

(a) $\displaystyle\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+5x) - 5x}$ (b) Compute $\displaystyle\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+5x) - 5x}$ **again** using series.

(c) $\displaystyle\lim_{x\to\infty} \left(\frac{x+1}{x}\right)^x$

2. [22 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{\cos x}{\left(4+\sin^2 x\right)^{\frac{5}{2}}}\,dx$ (b) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ (c) $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

3. [40 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_6^7 \frac{8}{x^2-4x-12}\,dx$ (b) $\displaystyle\int_7^\infty \frac{8}{x^2-4x-12}\,dx$ Tip: Reuse your algebra work from part (a)

(c) $\displaystyle\int_0^{e^3} \frac{1}{x\left[3+(\ln x)^2\right]}\,dx$ (d) $\displaystyle\int_0^1 \sqrt{x}\,\ln x\,dx$

4. [18 Points] Find the **sum** of each of the following series (which do converge). Simplify.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 5^{2n+1}}{2^{5n-1}}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(\ln 9)^n}{2^{n+1}\cdot n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n\,(2n)!}$

(d) $-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots$ (e) $-\frac{\pi^3}{3!}+\frac{\pi^5}{5!}-\frac{\pi^7}{7!}+\frac{\pi^9}{9!}-\dots$ (f) $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots$

5. [26 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(7n)}{n^7+7}$ (b) $\displaystyle\sum_{n=1}^{\infty} \arctan\left(\frac{n^7+1}{n^7+7}\right)$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{n^2}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3n)!\,\ln n}{(n!)^2\,e^{4n}\,n^n}$ (e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+5}{n^5+2}$

6. [18 Points] Find the **Interval** and **Radius** of Convergence for each of the following power series. Analyze carefully and with full justification.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\ (3x-4)^n}{n^2 \cdot 5^n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n}}{(2n)!}$ (c) $\displaystyle\sum_{n=1}^{\infty} n!\ (x-6)^n$

7. [10 Points] Please analyze with detail and justify carefully. Simplify.

(a) Use MacLaurin series to **Estimate** $\displaystyle\int_0^1 x\sin\left(x^2\right)\, dx$ with error less than $\dfrac{1}{1000}$.

(b) Use MacLaurin Series to **Estimate** $\dfrac{1}{\sqrt{e}}$ with error less than $\dfrac{1}{100}$.

8. [10 Points] Consider the region bounded by $y = \arctan x$, $y = 0$, $x = 0$ and $x = 1$. Rotate the region about the vertical line $\boxed{x = -1}$. **COMPUTE** the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

9. [18 Points]

(a) Consider the Parametric Curve represented by $x = \ln t + \ln(1-t^2)$ and $y = \sqrt{8}\arcsin t$.

COMPUTE the **arclength** of this parametric curve for $\dfrac{1}{4} \le t \le \dfrac{1}{2}$. **Show** that the answer simplifies to $\boxed{\ln\left(\dfrac{5}{2}\right)}$

(b) Consider a different Parametric Curve represented by $x = t - e^{2t}$ and $y = 1 - \sqrt{8}e^t$.

COMPUTE the **surface area** obtained by rotating this curve about the y-axis for $0 \le t \le 1$. Simplify. **Show** that the answer simplifies to $\boxed{2\pi\left(2 - \dfrac{e^4}{2}\right)}$

10. [20 Points] For each of the following problems, do the following **two** things:

1. Sketch the Polar curves and shade the described bounded region.

2. Set-Up but **DO NOT EVALUATE** the Integral representing the area of the described bounded region.

(a) The **area** bounded outside the polar curve $r = 1+\sin\theta$ and inside the polar curve $r = 3\sin\theta$.

(b) The **area** bounded outside the polar curve $r = 2$ and inside the polar curve $r = 4\sin\theta$.

(c) The **area** that lies inside both of the curves $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

(d) The **area** bounded inside the polar curve $r = 2 + 2\cos\theta$.