Math 121 Final Exam May 12, 2014

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 0} \frac{\cosh(4x) - 1 - \arctan(4x) + 4x}{\ln(1-x) + \arcsin x}$ (b) $\displaystyle\lim_{x\to\infty} \left[\sqrt{1-\frac{1}{x}} + \sinh\left(\frac{1}{x}\right)\right]^x$

2. [20 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{x^4 + 5x^2 - x + 3}{x^3 + 3x}\,dx$ (b) $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{\sqrt{36 - x^2}}\,dx$

(c) $\displaystyle\int_{-1}^{0} x^3\sqrt{1-x^2}\,dx$ **using a trigonometric substitution**

3. [30 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_7^{\infty} \frac{1}{x^2 - 8x + 19}\,dx$ (b) $\displaystyle\int_0^{e^5} \frac{1}{x[25 + (\ln x)^2]}\,dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \cosh x\,dx$ (d) $\displaystyle\int_0^1 \ln x\,dx$

4. [15 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 3^{n+2}}{2^{4n-1}}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, (\ln 8)^n}{3^{n+1}\, n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n+1)!}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$ (e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n)!}$

5. [35 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n + \sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{\ln(n+3)}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(7n)}{7^n} + \frac{7}{n^7 + 1}$ (e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^7 + 5}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, \pi^n\, (2n)!}{n^n\, (4)^n\, n!}$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the following power series $\sum_{n=0}^{\infty} \frac{(-1)^n\ (3x+2)^n}{(n+1)\ 4^n}$. Analyze carefully and with full justification.

7. [10 Points] (a) Write the **first 6 non-zero terms** of the MacLaurin Series for
$\boldsymbol{f(x) = \sin(x^3) + \cos(x^3)}$.

(b) Use this series to determine the **sixth**, **seventh**, **eighth** and **ninth** derivatives of
$\boldsymbol{f(x) = \sin(x^3) + \cos(x^3)}$ evaluated at $\boldsymbol{x = 0}$.

(Hint: Do not compute out those derivatives manually.)

(Hint: Write out the definition of the MacLaurin Series for any $\boldsymbol{f(x)}$.)

8. [15 Points] Please analyze with detail and justify carefully.

(a) Write the **MacLaurin series** representation for $\boldsymbol{f(x) = x \arctan\left(x^2\right)}$. Your answer should be in sigma notation $\boldsymbol{\sum_{n=0}^{\infty}}$.

(b) Use the MacLaurin series representation for $\boldsymbol{f(x) = x \arctan\left(x^2\right)}$ from Part(a) to

Estimate $\boldsymbol{\int_0^1 x \arctan\left(x^2\right)\, dx}$ with error less than $\boldsymbol{\frac{1}{50}}$. Justify in words that your error is indeed less than $\boldsymbol{\frac{1}{50}}$.

9. [15 Points]

(a) Consider the region bounded by $\boldsymbol{y = e^x + 2}$, $\boldsymbol{y = \sin x}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = \pi}$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = -2}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(b) Consider the region bounded by $\boldsymbol{y = \ln x}$, $\boldsymbol{y = 1}$, and $\boldsymbol{x = 4}$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = 5}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(c) Consider the region bounded by $\boldsymbol{y = \arctan x}$, $\boldsymbol{y = 0}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = 1}$. Rotate the region about the $\boxed{\boldsymbol{y}\text{-axis}}$. **COMPUTE** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [15 Points] Consider the Parametric Curve represented by $\boldsymbol{x = t - e^t}$ and $\boldsymbol{y = 1 - 4e^{\frac{t}{2}}}$.

(a) **COMPUTE** the **arclength** of this parametric curve for $\boldsymbol{0 \leq t \leq \ln 5}$.

(b) **Set-Up** but **DO NOT EVALUATE** the **surface area** obtained by rotating this same curve about the **$\boldsymbol{y}$-axis**, for $\boldsymbol{0 \leq t \leq 1}$.

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 2 + 2\sin\theta}$ and inside the polar curve $\boldsymbol{r = 6\sin\theta}$. **Sketch** the Polar curves.