Math 121 Final Exam May 11, 2012

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 0} \frac{3xe^x - \arctan(3x)}{x + \ln(1-x)}$ (b) $\displaystyle\lim_{x\to\infty} \left(\cosh\left(\frac{1}{x}\right) - \frac{5}{x}\right)^x$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{e^x}{(e^{2x}+4)^{\frac{5}{2}}}\,dx$ **Hint:** $e^{2x} = (e^x)^2$ (b) $\displaystyle\int x \arcsin x \, dx$ (c) $\displaystyle\int \frac{x^4 + 5x^2 - x + 3}{x^3 + 3x}\,dx$

3. [20 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_5^\infty \frac{1}{x^2 - 8x + 19}\,dx$ (b) $\displaystyle\int_0^9 \frac{1}{(x-1)^{\frac{4}{3}}}\,dx$

4. [15 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 3^{2n-1}}{4^{2n+1}}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{n!} = 1 - 2 + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} - \ldots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n)!}$

5. [30 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+7}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n+1}{n}\right)^n$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(3+n^2)}{n^7+4}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{5^n}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, \pi^n\, (3n)!}{n^n\, (27)^n\, (n!)^2}$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the power series $\sum_{n=0}^{\infty} \frac{(-1)^n\ (3x+4)^n}{(n^2+1)\ 5^n}$.
Analyze carefully and with full justification.

7. [10 Points] (a) Write the MacLaurin Series for $\boldsymbol{f(x) = x^5 \sin(x^3)}$.

(b) Use this series to determine the **eighth** and **ninth** derivatives of $f(x) = x^5 \sin(x^3)$ at $x = 0$.

(**Hint:** Do not compute out those derivatives manually.)

(**Hint:** Write out the definition of the MacLaurin Series for any $f(x)$.)

8. [15 Points] Please analyze with detail and justify carefully.

(a) Find the **MacLaurin series** representation for $\boldsymbol{f(x) = xe^{-x^7}}$. Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(b) Use the MacLaurin series representation for $\boldsymbol{f(x) = xe^{-x^7}}$ from Part(a) to

$$\textbf{Estimate} \quad \int_0^1 xe^{-x^7}\, dx \quad \text{with error less than } \frac{1}{10}.$$

Justify in words that your error is indeed less than $\frac{1}{10}$.

9. [15 Points] Consider the region bounded by $y = e^x$, $y = \ln x$, $x = 1$ and $x = 2$.

(a) Rotate the region about the line $\boxed{\boldsymbol{x = 8}}$. **Set-Up** but do **NOT** Evaluate the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(b) Rotate the region about the $\boxed{y\text{-axis}}$. **Compute** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [20 Points] Parametric Curves

(a) Consider the Parametric Curve represented by $\boldsymbol{x = \frac{t^3}{3} - \frac{e^{2t}}{2}}$ and $\boldsymbol{y = 2te^t - 2e^t}$.

Compute the **arclength** of this parametric curve for $0 \le t \le 1$.

(b) Consider the Parametric Curve represented by $\boldsymbol{x = \cos^3 t}$ and $\boldsymbol{y = \sin^3 t}$.

Compute the **surface area** obtained by rotating this curve about the y-**axis**, for $0 \le t \le \frac{\pi}{2}$.

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 2 + 2\cos\theta}$ and inside the polar curve $\boldsymbol{r = 6\cos\theta}$. **Sketch** the Polar curves.