



Math 121 Final December 19, 2024



- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.
- Numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\arctan(\sqrt{3})$ should be simplified.
- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [12 Points] (a) Use Series to show that $\lim_{x \rightarrow 0} \frac{x^2 + 4x - \arctan(4x)}{1 - 3x - e^{-3x}} = \boxed{-\frac{2}{9}}$

(b) Compute $\lim_{x \rightarrow 0} \frac{x^2 + 4x - \arctan(4x)}{1 - 3x - e^{-3x}}$ again using L'Hôpital's Rule.

2. [18 Points] Compute the following integrals. Simplify.

(a) $\int \frac{x^3}{(x^2 + 4)^{\frac{7}{2}}} dx$ Use Trig Sub and Hint: $\tan \theta = \frac{\sin \theta}{\cos \theta}$ (b) $\int x \arcsin x dx$

3. [28 Points] For each of the following **Improper** integrals, determine whether it Converges or Diverges. If it converges, find its value. Simplify.

(a) $\int_0^e \ln x dx$ (b) $\int_0^{e^4} \frac{4}{x[16 + (\ln x)^2]} dx$ (c) $\int_{-4}^{-3} \frac{8 - x}{x^2 + 2x - 8} dx$ (d) $\int_5^{\infty} \frac{7}{x^2 - 4x + 7} dx$

4. [24 Points] Find the **Sum** of each of the following series (which do converge). Simplify.

(a) $\frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \dots$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n (\ln 9)^n}{3! 2^n n!}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \pi^{2n}}{(36)^n (2n + 1)!}$

(d) $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \dots$ (e) $1 + 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \dots$ (f) Show that $\sum_{n=0}^{\infty} \frac{(-4)^n - 2}{5^n} = \boxed{-\frac{35}{18}}$

5. [22 Points] Find the **Interval** and **Radius** of Convergence for (a) and (b).

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n (4x + 1)^n}{(4n + 1)^2 7^n}$ (b) $\sum_{n=1}^{\infty} n^n (x - 7)^n$

(c) Design a Power Series centered at $a = 5$ which has which has an Infinite Radius of Convergence. Then proceed to justify that the Interval of Convergence is indeed $I = (-\infty, \infty)$.

6. [25 Points] Determine whether each series **Converges** or **Diverges**. Name any convergence test(s) you use, justify all of your work.

(a) $\sum_{n=2}^{\infty} \left(1 - \frac{7}{n^4}\right)^{n^4}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 + 7}$ Use Absolute Conv. Test (c) $\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$ Use Integral Test
 (d) $\sum_{n=2}^{\infty} \frac{n^4}{\ln n}$ (e) $-3 - \frac{3}{2} - 1 - \frac{3}{4} - \frac{3}{5} - \frac{1}{2} - \dots$ (f) $\sum_{n=1}^{\infty} \frac{1}{7^n} + \frac{4}{n^7}$

7. [18 Points] Determine whether each given series is **Absolutely Convergent** or **Conditionally Convergent**. Name any convergence test(s) you use, justify all of your work.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^4 + 7}{n^7 + 4}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n + 7}$

8. [7 Points] Use Series to Estimate $\int_0^1 x^3 \sin(x^2) dx$ with error less than $\frac{1}{10,000}$. Simplify.

Tips: $(120) \cdot (14) = 1680$ and $(60) \cdot (28) = 1680$ and $7! = 5040$ and $(5040) \cdot (18) = 90,720$

9. [20 Points] Answers in Sigma $\sum_{n=0}^{\infty}$ notation. **STATE** the Radius of Convergence for each.

(a) Demonstrate **TWO** different methods for deriving the MacLaurin Series for $\cos x$.

You may use the MacLaurin Series formula for $\sin x$ without extra justification.

(b) Find the MacLaurin Series Representation for $\ln(9 + x^2)$.

Use this formula: $\ln(9 + x^2) = \int \frac{2x}{9 + x^2} dx$. Yes, solve for C

(c) Show that the MacLaurin Series for $\frac{1}{2}(e^x - e^{-x})$ equals $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ Justify.

10. [8 Points] **COMPUTE** the Area bounded inside **Both** of the Polar curves

$r = 2 + 2 \cos \theta$ and $r = 2 - 2 \cos \theta$. Sketch the curves and shade the bounded region. HINT: Show that the described bounded area equals $\boxed{6\pi - 16}$

11. [18 Points] For **each** of the following problems, do the following **THREE** things:

1. **Sketch** the Polar curve(s) and **Shade** the described bounded region.

2. Set-Up but **DO NOT EVALUATE** an Integral representing the area of the described bounded region.

3. Set-Up but **DO NOT EVALUATE** another **slightly different** Integral representing the same area of the described bounded region.

(a) The **Area** bounded Outside the polar curve $r = 2 - 2 \sin \theta$ and Inside $r = -6 \sin \theta$.

(b) The **Area** bounded Inside **Both** of the curves $r = 3 + 3 \sin \theta$ and $r = 3 - 3 \sin \theta$.

(c) The **Area** bounded Inside **Both** of the polar curves $r = 2 \sin \theta$ and $r = -2 \cos \theta$.