Math 121 Final Exam November 16-20, 2020

Due Friday, November 20, in Gradescope by 11:59 pm EST

- This is an *Open Notes* Exam. You can use materials, homework problems, lecture notes, etc. that you manually worked on.
- There is **NO** *Open Internet* allowed. You can only access our Main Course Webpage.
- You are not allowed to work on or discuss these problems with anyone. You can ask me a few small, clarifying, questions about instructions in Office Hours, but these problems will not be solved.
- Submit your final work in Gradescope in the **Final Exam** entry.
- Please *show* all of your work and *justify* all of your answers.

1. [10 Points] Show that $\lim_{x\to\infty}\left(\dfrac{x}{x+1}\right)^x = \boxed{\dfrac{1}{e}}$

2. [20 Points] Evaluate the following **definite integrals**.

(a) Show that $\displaystyle\int_{-1}^{0} x^4 \arcsin x \; dx = \boxed{\frac{8}{75} - \frac{\pi}{10}}$

(b) Show that $\displaystyle\int_{-2}^{2} \sqrt{4-x^2} \; dx = \boxed{2\pi}$

3. [30 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value. Simplify.

(a) $\displaystyle\int_{-\infty}^{0} \frac{1}{x^2+2x+4} \; dx$

(b) $\displaystyle\int_{-4}^{-3} \frac{6}{x^2+2x-8} \; dx$

(c) $\displaystyle\int_{0}^{e} x^2 \ln\left(x^2\right) \; dx$

You can use this **free** given P.F.D. fact:

$$\frac{6}{(x-2)(x+4)} = \frac{1}{x-2} - \frac{1}{x+4}$$

4. [24 Points] Find the **sum** of each of the following series (which do converge). Simplify.

(a) $\dfrac{1}{2!} - \dfrac{1}{3!} + \dfrac{1}{4!} - \dfrac{1}{5!} + \dots$

(b) $2 - 1 + \dfrac{2}{3} - \dfrac{2}{4} + \dfrac{2}{5} - \dots$

(c) $-\pi + \dfrac{\pi^3}{3!} - \dfrac{\pi^5}{5!} + \dfrac{\pi^7}{7!} - \dfrac{\pi^9}{9!} + \dots$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \; \pi^{2n+1}}{(\sqrt{2})^{4n} \; (2n)!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \; e^n}{n!}$

(f) $1 + 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dfrac{1}{9} - \dots$

5. [20 Points] In each case determine whether the given series is **Convergent** or **Divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(n^8+8\right)$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^8+8}{n^8+1}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^8+1}$

6. [16 Points] Find the **Interval** and **Radius** of Convergence for the following power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n\ (6x+1)^n}{(6n+1)\ 7^n}$$

Analyze carefully and with full justification.

7. [10 Points] Use MacLaurin Series to **Estimate** $\ln\left(\dfrac{3}{2}\right)$ with error less than $\dfrac{1}{50}$. Simplify.

Hint: $\ln\left(\dfrac{3}{2}\right)=\ln\left(1+\dfrac{1}{2}\right)$

8. [10 Points] Find the MacLaurin Series for $F(x)=\dfrac{1}{(1+3x)^2}$ and **State** the Radius of Convergence. Your answer should be in Sigma notation $\displaystyle\sum_{n=0}^{\infty}$ here.

Hint: $\dfrac{1}{(1+3x)^2}=\dfrac{d}{dx}\left[-\dfrac{1}{3(1+3x)}\right]$

9. [10 Points] Your answers should be in Sigma notation $\displaystyle\sum_{n=0}^{\infty}$. You do **not** need the Radius.

(a) Find the MacLaurin Series for $G(x)=x^2\arctan(2x^3)$.

(b) Use the Series found in part (a) to compute $\displaystyle\int x^2\arctan(2x^3)\ dx$

See next page for **OPTIONAL** BONUS Questions

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OPTIONAL BONUS

Do not attempt these unless you are completely done with the rest of the exam.

To get Bonus Points, you must present a complete, correct, solution for the entire problem.
Please submit full solutions only.

You can attempt any of them; you do not need to do all of them.
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OPTIONAL BONUS #1 [+2 Points] Use Integration to demonstrate a different way to compute the Series for $G(x) = x^2 \arctan(2x^3)$ in 9(a) above.

OPTIONAL BONUS #2 [+2 Points] Compute $\displaystyle\lim_{x \to 0} \frac{4xe^x - \arctan(4x)}{\sin(2x) + \ln(1-2x)}$ using two different methods:

(a) Use L'Hôpital's Rule

(b) Use Series.

OPTIONAL BONUS #3 [+3 Points] Compute the value of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(3n+2)}{(n+1)(2n+1)\ 3^{n+1}}$ by first computing $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(3n+2)\ x^{2n+2}}{(n+1)(2n+1)}$.

OPTIONAL BONUS #4 [+2 Points] Compute the Area that lies outside the Polar Curve $r = 2 + 2\sin\theta$ and inside the Polar Curve $r = 6\sin\theta$.

Sketch the Polar Curves and shade the described bounded region.

OPTIONAL BONUS #5 [+3 Points] Compute the Area that lies inside both of the Polar Curves $r = 2 + 2\cos\theta$ and $r = 2 - 2\sin\theta$.

Sketch the Polar Curves and shade the described bounded region.