Math 121 Final Exam December 19, 2014

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, $\arctan(\sqrt{3})$, or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim\limits_{x\to 0} \dfrac{\arctan(3x) - e^{3x} + 1}{x^2}$ (b) $\lim\limits_{x\to 0} \dfrac{\sinh(3x) - 3x}{\ln(1+x) - x}$ (c) $\lim\limits_{x\to\infty} \left[1 + \ln\left(1 + \dfrac{5}{x}\right)\right]^x$

2. [20 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int x \arcsin x \; dx$ (b) $\displaystyle\int_4^{4\sqrt{3}} \frac{1}{\sqrt{64 - x^2}} + \frac{1}{16 + x^2} \; dx$ (c) $\displaystyle\int_1^e \frac{1}{x\,[1 + (\ln x)^2]^{\frac{3}{2}}} \; dx$

3. [30 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_9^\infty \frac{1}{x^2 - 8x + 41} \; dx$ (b) $\displaystyle\int_1^2 \frac{1}{x \ln x} \; dx$ (c) $\displaystyle\int_0^\infty \frac{1}{x^2 + 3x + 2} \; dx$ (d) $\displaystyle\int_0^1 \ln x \; dx$

4. [15 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n \; 6^{n+1}}{2^{3n-1}}$ (b) $\displaystyle\sum_{n=0}^\infty \frac{2^{n+1} \; (\ln 3)^n}{n!}$ (c) $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n \; \pi^{2n}}{(16)^n \; (2n+1)!}$

(d) $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$ (e) $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

5. [35 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **divergent**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n \; n}{n^2 + 1}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{\ln n}{n^2 + 1}$ (c) $\displaystyle\sum_{n=1}^\infty \arctan n$

(d) $\displaystyle\sum_{n=1}^\infty \frac{\sin^2(7n)}{7^n} + \frac{7}{n^7}$ (e) $\displaystyle\sum_{n=1}^\infty (-1)^n \; \frac{\sqrt{n} + 8}{n^3 + 2}$ (f) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n \; (3n)!}{(n!)^2 \; n^n \; e^{4n}}$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n\ (5x-4)^n}{n^3\ 9^n}$. Analyze carefully and with full justification.

7. [10 Points]

(a) Write the **first 4 non-zero terms** of the MacLaurin Series for $\boldsymbol{f(x) = \arctan(x^2)}$.

(b) Write the **first 4 non-zero terms** of the MacLaurin Series for the hyperbolic cosine $\boldsymbol{f(x) = \cosh x}$.

(c) Write the **first 4 non-zero terms** of the MacLaurin Series for $\boldsymbol{f(x) = \cosh(x^2)}$.

(d) Write the **first 6 non-zero terms** of the MacLaurin Series for $\boldsymbol{f(x) = \arctan(x^2) + \cosh(x^2)}$.

(e) Use this series to determine the **sixth**, **seventh**, **eighth** and **ninth** derivatives of $\boldsymbol{f(x) = \arctan(x^2) + \cosh(x^2)}$ evaluated at $\boldsymbol{x = 0}$. Simplify your answers.

8. [15 Points] Please analyze with detail and justify carefully.

(a) Write the **MacLaurin series** representation for $\boldsymbol{f(x) = x^3 e^{-x^2}}$. Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(b) Use the MacLaurin series representation for $\boldsymbol{f(x) = x^3 e^{-x^2}}$ from Part(a) to **Estimate** $\int_0^1 x^3 e^{-x^2}\, dx$ with error less than $\frac{1}{10}$. Justify in words that your error is less than $\frac{1}{10}$.

9. [15 Points]

(a) Consider the region bounded by $\boldsymbol{y = e^x + 2}$, $\boldsymbol{y = \sin x}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = \pi}$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = -2}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(b) Consider the region bounded by $\boldsymbol{y = \ln x}$, $\boldsymbol{y = 1}$, and $\boldsymbol{x = 4}$. Rotate the region about the vertical line $\boxed{\boldsymbol{x = 5}}$. **Set-Up** but **DO NOT EVALUATE** the integral representing the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

(c) Consider the region bounded by $\boldsymbol{y = \arctan x}$, $\boldsymbol{y = 4}$, $\boldsymbol{x = 0}$ and $\boldsymbol{x = 1}$. Rotate the region about the $\boxed{\boldsymbol{y}\text{-axis}}$. **COMPUTE** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [15 Points] Consider the Parametric Curve represented by $\boldsymbol{x = e^t - t}$ and $\boldsymbol{y = 1 - 4e^{\frac{t}{2}}}$.

(a) **COMPUTE** the **arclength** of this parametric curve for $\boldsymbol{0 \le t \le 1}$.

(b) **COMPUTE** the **surface area** obtained by rotating this same curve about the $\boldsymbol{y}$**-axis**, for $\boldsymbol{0 \le t \le 1}$.

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 2 + 2\cos\theta}$ and inside the polar curve $\boldsymbol{r = 6\cos\theta}$. **Sketch** the Polar curves.