Math 121 Final Exam December 18, 2012

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $\arctan(\sqrt{3})$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\cosh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 0} \frac{\ln(1+x) - \sinh x}{\arctan(2x) - e^{2x} + 1}$ (b) $\displaystyle\lim_{x\to\infty} \left(1 - \frac{2}{x^3}\right)^{7x^3}$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$ (b) $\displaystyle\int x \arcsin x\,dx$ (c) $\displaystyle\int \frac{x^4+4x^2+x+6}{x^3+2x}\,dx$

3. [20 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_3^{\infty} \frac{1}{x^2-4x+7}\,dx$ (b) $\displaystyle\int_0^1 \ln x\,dx$

4. [15 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, 5^{n+1}}{3^{2n-1}}$ (b) $1 - \ln 5 + \dfrac{(\ln 5)^2}{2!} - \dfrac{(\ln 5)^3}{3!} + \dfrac{(\ln 5)^4}{4!} - \dots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(36)^n (2n+1)!}$

5. [30 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{7n+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\, \pi^n\, (3n)!}{n^n\, (27)^n\, (n!)^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2+3}{n^7+4}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{5^n} + \frac{5}{n^5}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

6. [15 Points] Find the **Interval** and **Radius** of Convergence for the following power series $\sum_{n=1}^{\infty}(-1)^n \frac{3^n(x+1)^n}{n^4\sqrt{n}}$. Analyze carefully and with full justification.

7. [10 Points] (a) Write the MacLaurin Series for $f(x) = x\ln(1+x^2)$.

(b) Use this series to determine the **fifth** and **sixth** derivatives of $f(x) = x\ln(1+x^2)$ at $x = 0$.

(**Hint:** Do not compute out those derivatives manually.)

(**Hint:** Write out the definition of the MacLaurin Series for any $f(x)$.)

8. [15 Points] Please analyze with detail and justify carefully.

(a) Find the **MacLaurin series** representation for $\boldsymbol{f(x) = x^2 \sin(x^2)}$.

Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(b) Use the MacLaurin series representation for $\boldsymbol{f(x) = x^2 \sin(x^2)}$ from Part(a) to

$$\textbf{Estimate} \quad \int_0^1 x^2 \sin(x^2)\, dx \quad \text{with error less than } \frac{1}{10}.$$

Justify in words that your error is indeed less than $\frac{1}{10}$.

9. [15 Points] Consider the region bounded by $y = \sin x$, and $y = 0$, between $x = 0$ and $x = \frac{\pi}{2}$. Rotate the region about the vertical line $\boxed{x = 8}$. **Compute** the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [20 Points] Parametric Curves

(a) Consider the Parametric Curve represented by $\boldsymbol{x = e^t \cos t}$ and $\boldsymbol{y = e^t \sin t}$.

Compute the **arclength** of this parametric curve for $0 \le t \le \pi$.

(b) Consider a different Parametric Curve represented by $\boldsymbol{x = 3 - 2t}$ and $\boldsymbol{y = e^t + e^{-t}}$.

Compute the **surface area** obtained by rotating this curve about the x-**axis**, for $0 \le t \le 1$.

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 2 + 2\cos\theta}$ and inside the polar curve $\boldsymbol{r = 6\cos\theta}$. **Sketch** the Polar curves.