Math 121 Final Exam December 21, 2011

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, $e^{3\ln 3}$, or $\sinh(\ln 3)$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [15 Points] Evaluate each of the following **limits**. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to\ln 3} \frac{1}{e^x-3} - \frac{1}{3x-\ln 27}$ (b) $\displaystyle\lim_{x\to\infty} \left(\cosh\left(\frac{1}{x}\right) - \frac{5}{x}\right)^x$

2. [30 Points] Evaluate each of the following **integrals**.

(a) $\displaystyle\int \frac{1}{(x^2+4)^{\frac{7}{2}}} dx$ (b) $\displaystyle\int x \ \arcsin x \ dx$ (c) $\displaystyle\int \frac{x^4+x^3+4x^2+5x+4}{x^3+4x} \ dx$

3. [20 Points] For each of the following **improper integrals**, determine whether it converges or diverges. If it converges, find its value.

(a) $\displaystyle\int_3^\infty \frac{1}{x^2-4x+7} \ dx$ (b) $\displaystyle\int_0^1 \frac{e^{\frac{1}{x}}}{x^2} \ dx$

4. [10 Points] Find the **sum** of each of the following series (which do converge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n+2}}{2^{4n-1}}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots$ (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n)!}$

5. [25 Points] In each case determine whether the given series is **absolutely convergent**, **conditionally convergent**, or **diverges**. Justify your answers.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (9+n^2)}{n^7+1}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ \pi^n \ (3n)!}{n^n \ (27)^n \ (n!)^2}$

6. [10 Points] Find the **Interval** and **Radius** of Convergence for the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(4x+1)^n}{(n^2+1)\,7^n}$. Analyze carefully and with full justification.

7. [5 Points] Write the MacLaurin Series for $\boldsymbol{f(x) = x^5 \sin(x^3)}$. Use this series to determine the **eighth** and **ninth** derivatives of $f(x)$ at $x = 0$. (Do not compute out those derivatives manually.)

8. [10 Points] Please analyze with detail and justify carefully.

(a) Find the **MacLaurin series** representation for $\boldsymbol{f(x) = xe^{-x^7}}$. Your answer should be in sigma notation $\displaystyle\sum_{n=0}^{\infty}$.

(b) Use the MacLaurin series representation for $\boldsymbol{f(x) = xe^{-x^7}}$ from Part(a) to

$$\textbf{Estimate} \quad \int_0^1 xe^{-x^7}\,dx \quad \text{with error less than } \frac{1}{10}.$$

Justify in words that your error is indeed less than $\frac{1}{10}$.

9. [15 Points] Consider the region bounded by $\boldsymbol{y = 1}$, $\boldsymbol{y = \ln x}$, $\boldsymbol{x = 1}$. Rotate the region about the line $\boxed{\boldsymbol{x = -1}}$. **Compute** the **volume** of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating cylindrical shells.

10. [15 Points] Parametric Curves

(a) Consider the Parametric Curve represented by $\boldsymbol{x = \frac{t^3}{3} - \frac{e^{2t}}{2}}$ and $\boldsymbol{y = 2te^t - 2e^t}$. Compute the **arclength** of this parametric curve for $0 \le t \le 1$.

(b) Consider the Parametric Curve represented by $\boldsymbol{x = \cos^3 t}$ and $\boldsymbol{y = \sin^3 t}$. Compute the **surface area** obtained by rotating this curve about the y-**axis**, for $0 \le t \le \frac{\pi}{2}$.

11. [15 Points] Compute the **area** bounded outside the polar curve $\boldsymbol{r = 2 + 2\cos\theta}$ and inside the polar curve $\boldsymbol{r = 6\cos\theta}$. **Sketch** the Polar curves.

12. [10 Points] Find the general solution for each of the following **differential equations**.

(a) $\dfrac{dy}{dx} = \dfrac{e^y}{\sqrt{1-x^2}}$

(b) $x\,\dfrac{dy}{dx} = x^7\cosh x + 4y$