Math 121 Self-Assessment Practice Worksheet #9 Answer Key

Find the Interval and Radius of Convergence for the folllowing Power Series. Analyze carefully and with full justification.

Note: After Exam 2, we (in Benedetto's class) are no longer required to justify the limiting values at infinity for stacks of polynomials as settling to the stacked coefficents of the highest powered terms.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\,(6x+1)^n}{(6n+1)\,7^n}$

Use Ratio Test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,(6x+1)^{n+1}}{(6(n+1)+1)\,7^{n+1}}}{\dfrac{\cancel{(-1)^{n}}\,(6x+1)^{n}}{(6n+1)\,7^{n}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(6x+1)^{n+1}}{(6x+1)^n}\right|\cdot\cancelto{1}{\left(\frac{6n+1}{6n+7}\right)}\cdot\frac{7^n}{7^{n+1}} = \frac{|6x+1|}{7}$$

The Ratio Test gives convergence for x when $\dfrac{|6x+1|}{7} < 1$ or $|6x+1| < 7$.

That is $-7 < 6x+1 < 7 \Longrightarrow -8 < 6x < 6 \Longrightarrow -\dfrac{4}{3} < x < 1$

Endpoints:

• $x = 1$ The original series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,(6(1)+1)^n}{(6n+1)\,7^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\,\cancel{7^n}}{(6n+1)\,\cancel{7^n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{6n+1}$ which is itself Convergent by the Alternating Series Test:

1. $b_n = \dfrac{1}{6n+1} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\cancelto{0}{\frac{1}{\cancelto{\infty}{6n+1}}} = 0$

3. Terms are Decreasing: $b_{n+1} < b_n$ because $b_{n+1} = \dfrac{1}{6(n+1)+1} = \dfrac{1}{6n+7} < \dfrac{1}{6n+1} = b_n$.

OR the related function $f(x) = \dfrac{1}{6x+1}$ has derivative $f'(x) = -\dfrac{6}{(6x+1)^2} < 0$ so the terms are decreasing.

$\bullet x = -\frac{4}{3}$ The original series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n \left(6\left(-\frac{4}{3}\right)+1\right)^n}{(6n+1)\ 7^n} = \sum_{n=0}^{\infty} \frac{(-1)^n\ (-7)^n}{(6n+1)\ 7^n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n\ \cancel{7^n}}{(6n+1)\ \cancel{7^n}} = \sum_{n=0}^{\infty} \frac{\cancel{(-1)^{2n}}^{\,1}_{\text{even}}}{6n+1} = \sum_{n=0}^{\infty} \frac{1}{6n+1}$

Here $\sum_{n=0}^{\infty} \frac{1}{6n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ the divergent Harmonic p-Series, $p = 1$.

LCT Limit: $\lim_{n\to\infty} \frac{\frac{1}{6n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{6n+1} = \frac{1}{6}$ which is *Finite* and *Non-zero*.

Therefore, $\sum_{n=0}^{\infty} \frac{1}{6n+1}$ is also Divergent by LCT.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{4}{3}, 1\right]}$ with Radius of Convergence $\boxed{R = \frac{7}{6}}$.

2. $\sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n+1}}{(2n+1)!}$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}}\ x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{\cancel{(-1)^n} x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{x^{2n+1}}\right| \cdot \frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty} |x|^2 \cdot \frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}}$$

$$= \lim_{n\to\infty} \frac{|x|^2 \to 0}{(2n+3)(2n+2) \to \infty} = 0 < 1$$

Here, the Series Converges by the Ratio Test for all Real numbers x.

Finally, Interval of Convergence $\boxed{I = (-\infty, \infty)}$ with Radius of Convergence $\boxed{R = \infty}$.

Note: this is the Power Series for $\sin x$, and this is confirmation that sine's Power Series has Infinite Radius of Convergence.

3. $\displaystyle\sum_{n=0}^{\infty} n!\,(x-9)^n$

Use Ratio Test:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-9)^{n+1}}{n!\,(x-9)^n}\right| = \lim_{n\to\infty}|x-9|\cdot\frac{(n+1)\cancel{n!}}{\cancel{n!}}$$

$$= \lim_{n\to\infty}|x-9|\,\cancelto{\infty}{(n+1)} = \infty > 1$$

The Series Diverges by the Ratio Test for all x *unless* $x = 9$ (where $L = 0 < 1$).

Finally, Interval of Convergence $\boxed{I = \{9\}}$ with Radius of Convergence $\boxed{R = 0}$.

4. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,(7x-3)^n}{(n+1)\,5^n}$

Use Ratio Test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{\cancel{(-1)^{n+1}}\,(7x-3)^{n+1}}{(n+2)\,5^{n+1}}}{\dfrac{\cancel{(-1)^n}\,(7x-3)^n}{(n+1)\,5^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(7x-3)^{n+1}}{(7x-3)^n}\right|\cdot\left(\cancelto{1}{\frac{n+1}{n+2}}\right)\cdot\frac{5^n}{5^{n+1}} = \frac{|7x-3|}{5}$$

The Ratio Test gives convergence for x when $\dfrac{|7x-3|}{5} < 1$ or $|7x-3| < 5$.

That is $-5 < 7x-3 < 5 \Longrightarrow -2 < 7x < 8 \Longrightarrow -\dfrac{2}{7} < x < \dfrac{8}{7}$

Endpoints:

• $x = \dfrac{8}{7}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\left(7\left(\frac{8}{7}\right)-3\right)^n}{(n+1)\,5^n} = \sum_{n=0}^{\infty}\frac{(-1)^n\,\cancel{5^n}}{(n+1)\,\cancel{5^n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$
which is itself Convergent by the Alternating Series Test:

1. $b_n = \dfrac{1}{n+1} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\cancelto{0}{\frac{1}{\cancelto{\infty}{n+1}}} = 0$

3. Terms are Decreasing: $b_{n+1} < b_n$ because $b_{n+1} = \dfrac{1}{n+2} < \dfrac{1}{n+1} = b_n$.

OR the related funtion $f(x) = \dfrac{1}{x+1}$ has derivative $f'(x) = -\dfrac{1}{(x+1)^2} < 0$ so the terms are decreasing.

$\bullet x = -\dfrac{2}{7}$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(7\left(-\frac{2}{7}\right) - 3\right)^n}{(n+1)\ 5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-5)^n}{(n+1)\ 5^n}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n\ \cancel{5^n}}{(n+1)\ \cancel{5^n}} = \sum_{n=0}^{\infty} \frac{\cancel{(-1)^{2n}}^{\,1} \text{even}}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

Here

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$ the divergent Harmonic p-Series, $p = 1$.

LCT: $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1$ which is *Finite* and *Non-zero*.

Therefore, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ is also Divergent by LCT.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{2}{7}, \frac{8}{7}\right]}$ with Radius of Convergence $\boxed{R = \frac{5}{7}}$.

5. $\displaystyle\sum_{n=0}^{\infty} n^n(3x+5)^n$

Use Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^{n+1}\ (3x+5)^{n+1}}{n^n\ (3x+5)^n}\right| = \lim_{n\to\infty} |3x+5| \cdot \frac{(n+1)^n(n+1)}{n^n}$$

$$= \lim_{n\to\infty} |3x+5| \cdot \cancel{\frac{(n+1)^n}{n^n}}^{\,e} \cdot \cancel{(n+1)}^{\,\infty} = \infty > 1$$

The Series Diverges by the Ratio Test for all x *unless* $3x+5=0$ or when $x = -\dfrac{5}{3}$.

Finally, Interval of Convergence $\boxed{I = \left\{-\frac{5}{3}\right\}}$ with Radius of Convergence $\boxed{R = 0}$.

6. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Use Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{x^{n+1}}{(n+1)!}}{\dfrac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} |x| \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty} \frac{|x|}{\cancel{(n+1)}} \overset{0}{\underset{\infty}{=}} 0 < 1$$

Here, the Series Converges by the Ratio Test for all Real numbers x.

Finally, Interval of Convergence $\boxed{I = (-\infty, \infty)}$ with Radius of Convergence $\boxed{R = \infty}$.

Note: this is the Power Series for e^x, and this is confirmation that Exponential's Power Series has Infinite Radius of Convergence.

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ (x+5)^n}{(n+3)^8\ 4^n}$ Use Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{\cancel{(-1)^{n+1}}(x+5)^{n+1}}{((n+1)+3)^8 4^{n+1}}}{\dfrac{\cancel{(-1)^n}(x+5)^n}{(n+3)^8 4^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{(x+5)^n} \right| \cdot \left(\frac{\cancel{n+3}}{\cancel{n+4}} \right)^{8} \cdot \frac{4^n}{4^{n+1}} = \frac{|x+5|}{4}$$

The Ratio Test gives convergence for x when $\dfrac{|x+5|}{4} < 1$ or $|x+5| < 4$

That is $-4 < x+5 < 4 \Longrightarrow -9 < x < -1$

Endpoints:

•$x = -9$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ (-9+5)^n}{(n+3)^8\ 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n(-4)^n}{(n+3)^8\ 4^n}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n \cancel{4^n}}{(n+3)^8 \cancel{4^n}} = \sum_{n=0}^{\infty} \frac{\cancel{(-1)^{2n}}^{\,1}}{(n+3)^8} = \sum_{n=0}^{\infty} \frac{1}{(n+3)^8}$$

Here

$$\sum_{n=0}^{\infty} \frac{1}{(n+3)^8} \approx \sum_{n=1}^{\infty} \frac{1}{n^8} \text{ the Convergent } p\text{-Series, } p = 8 > 1.$$

CT Bound Terms: $\dfrac{1}{(n+3)^8} < \dfrac{1}{n^8}$

Therefore, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+3)^8}$ is also Convergent by the (Direct) Comparison Test.

Note: The Limit Comparison Test also works here on this endpoint.

•$x = -1$ The original series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\,(-1+5)^n}{(n+3)^8\ 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n\,\cancel{4^n}}{(n+3)^8\ \cancel{4^n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+3)^8}$

Consider the Absolute Series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+3)^8}$ which was shown above (with the other endpoint) to be Convergent using the Comparison Test.

Therefore, THIS endpoint Converges using the Absolute Convergence Test.

Note: You can also use the Alternating Series Test here for this endpoint $x = -1$.

Finally, Interval of Convergence $\boxed{I = [-9, -1]}$ with Radius of Convergence $\boxed{R = 4}$.