1. $$\int \frac{1}{x^2-8x+25}\,dx = \int \frac{1}{(x-4)^2+9}\,dx \quad \text{complete the square}$$
$$= \int \frac{1}{u^2+9}\,du$$
$$= \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C$$
$$= \boxed{\frac{1}{3}\arctan\left(\frac{x-4}{3}\right) + C}$$

Sub: $\boxed{\begin{aligned} u &= x-4 \\ du &= dx \end{aligned}}$

2. $$\int \frac{10}{x^2-8x-9}\,dx = \int \frac{10}{(x-9)(x+1)}\,dx$$
$$\overset{\text{PFD}}{=} \int \frac{1}{x-9} - \frac{1}{x+1}\,dx$$
$$= \boxed{\ln|x-9| - \ln|x+1| + C}$$

Partial Fractions Decomposition:

$$\frac{10}{(x-9)(x+1)} = \frac{A}{x-9} + \frac{B}{x+1}$$

Clearing the denominator yields:

$10 = A(x+1) + B(x-9)$
$10 = (A+B)x + A - 9B$
so that $A+B=0$, and $A-9B=10$
Solve for $A=1$ and $B=-1$

$$3. \int \frac{x+7}{x^2+6x+14}\,dx = \int \frac{x+7}{(x+3)^2+5}\,dx \quad \overset{\text{complete the}}{\text{square}}$$

$$= \int \frac{(u-3)+7}{u^2+5}\,du$$

$$= \int \frac{u+4}{u^2+5}\,du$$

$$= \int \frac{u}{u^2+5} + \frac{4}{u^2+5}\,du$$

$$= \frac{1}{2}\int \frac{1}{w}\,dw + \frac{4}{\sqrt{5}}\arctan\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= \frac{1}{2}\ln|w| + \frac{4}{\sqrt{5}}\arctan\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= \frac{1}{2}\ln|u^2+5| + \frac{4}{\sqrt{5}}\arctan\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= \boxed{\frac{1}{2}\ln|(x+3)^2+5| + \frac{4}{\sqrt{5}}\arctan\left(\frac{x+3}{\sqrt{5}}\right) + C}$$

Sub: $\boxed{\begin{aligned} u &= x+3 \Rightarrow x = u-3 \\ du &= dx \end{aligned}}$

Sub: $\boxed{\begin{aligned} w &= u^2+5 \\ dw &= 2udu \\ \tfrac{1}{2}dw &= udu \end{aligned}}$

$$4. \int \frac{5x^2-13x+16}{x^3-3x^2+2x-6}\,dx = \int \frac{5x^2-13x+16}{(x-3)(x^2+2)}\,dx$$

$$\overset{\text{PFD}}{=} \int \frac{2}{x-3} + \frac{3x-4}{x^2+2}\,dx$$

$$= \int \frac{2}{x-3} + \frac{3x}{x^2+2} - \frac{4}{x^2+2}\,dx$$

$$= 2\ln|x-3| + \frac{3}{2}\ln|x^2+2| - \frac{4}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C$$

Partial Fractions Decomposition:

$$\frac{5x^2-13x+16}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$$

Clearing the denominator yields:

$5x^2 - 13x + 6 = A(x^2+2) + (Bx+C)(x-3)$
$5x^2 - 13x + 6 = (A+B)x^2 + (C-3B)x + 2A - 3C$
so that $A+B=5,\ C-3B=-13$ and $2A-3C=16$
Solve for $A=2,\ B=3$ and $C=-4$

5. $$\int_2^\infty \frac{x}{e^{3x}}\,dx = \lim_{t\to\infty}\int_2^t \frac{x}{e^{3x}}\,dx \quad \text{set-up improper limit right away}$$

$$= \lim_{t\to\infty}\int_2^t xe^{-3x}\,dx$$

$$\overset{\text{IBP}}{=} \lim_{t\to\infty} -\frac{x}{3e^{3x}}\Bigg|_2^t - \int_2^t -\frac{1}{3}e^{-3x}\,dx$$

$$= \lim_{t\to\infty} -\frac{x}{3e^{3x}}\Bigg|_2^t + \frac{1}{3}\int_2^t e^{-3x}\,dx$$

$$= \lim_{t\to\infty} -\frac{x}{3e^{3x}}\Bigg|_2^t - \frac{e^{-3x}}{9}\Bigg|_2^t$$

$$= \lim_{t\to\infty} -\frac{x}{3e^{3x}}\Bigg|_2^t - \frac{1}{9e^{3x}}\Bigg|_2^t$$

$$\lim_{t\to\infty} -\frac{t}{3e^{3t}}\overset{\frac{\infty}{\infty}}{} - \left(-\frac{2}{3e^6}\right) - \left(\cancelto{0}{\frac{1}{9e^{3t}}}{}^{\infty} - \frac{1}{9e^6}\right)$$

$$\overset{\text{L'H}}{=} \lim_{t\to\infty} -\cancelto{0}{\frac{1}{9e^{3t}}}{}^{\infty} + \frac{2}{3e^6} - 0 + \frac{1}{9e^6}$$

$$= 0 + \frac{2}{3e^6} + \frac{1}{9e^6} = \frac{6}{9e^6} + \frac{1}{9e^6} = \boxed{\frac{7}{9e^6}}$$

IBP: $$\boxed{\begin{array}{ll} u = x & dv = e^{-3x}dx \\ du = dx & v = -\frac{1}{3}e^{-3x} \end{array}}$$

6. $$\int_1^\infty \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{e^{\frac{1}{x}}}{x^2}\,dx = \lim_{t\to\infty} -\int_1^{\frac{1}{t}} e^u\,du$$

$$= \lim_{t\to\infty} -e^u\Big|_1^{\frac{1}{t}} = \lim_{t\to\infty} -e^{\frac{1}{t}} + e^1 = \lim_{t\to\infty} -\overset{1}{\cancel{e^{\overset{0}{\cancel{\frac{1}{t}}}}}} + e = \boxed{e-1}$$

Sub: $$\boxed{\begin{aligned} u &= \frac{1}{x} \\ du &= -\frac{1}{x^2}dx \\ -du &= \frac{1}{x^2}dx \end{aligned}} \qquad \boxed{\begin{aligned} x = 1 &\Rightarrow u = 1 \\ x = t &\Rightarrow u = \frac{1}{t} \end{aligned}}$$

7. $$\int_8^\infty \frac{1}{x^2-10x+28}\,dx = \lim_{t\to\infty}\int_8^t \frac{1}{(x-5)^2+3}\,dx$$ complete the square

$$= \lim_{t\to\infty}\int_3^{t-5} \frac{1}{u^2+3}\,du = \lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_3^{t-5}$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\arctan\left(\frac{t-5}{\sqrt{3}}\right) - \arctan\left(\frac{3}{\sqrt{3}}\right)\right)$$

$$= \lim_{t\to\infty}\frac{1}{\sqrt{3}}\left(\overset{\frac{\pi}{2}}{\cancel{\arctan\left(\frac{\overset{\infty}{\cancel{t-5}}}{\sqrt{3}}\right)}} - \arctan\left(\sqrt{3}\right)\right)$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(\frac{3\pi}{6}-\frac{2\pi}{6}\right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$

Substitute $$\boxed{\begin{aligned} u &= x-5 \\ du &= dx \end{aligned}} \qquad \boxed{\begin{aligned} x = 8 &\Rightarrow u = 3 \\ x = t &\Rightarrow u = t-5 \end{aligned}}$$

$$8.\quad \int_{-\infty}^{0} \frac{6}{x^2+2x+4}\,dx = \lim_{s\to-\infty} \int_{s}^{0} \frac{6}{x^2+2x+4}\,dx$$

$$= \lim_{s\to-\infty} \int_{s}^{0} \frac{6}{(x+1)^2+3}\,dx$$

$$= \lim_{s\to-\infty} \int_{s+1}^{1} \frac{6}{u^2+3}\,dx$$

$$= \lim_{s\to-\infty} \frac{6}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{s+1}^{1}$$

$$= \lim_{s\to-\infty} \frac{6}{\sqrt{3}} \left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \overset{-\frac{\pi}{2}}{\arctan\left(\frac{\overset{-\infty}{s}+1}{\sqrt{3}}\right)} \right)$$

$$= \frac{6}{\sqrt{3}}\left[\frac{\pi}{6} - \left(-\frac{\pi}{2}\right)\right] = \frac{6}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{2}\right)$$

$$= \frac{6}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{3\pi}{6}\right) = \frac{\cancel{6}}{\sqrt{3}}\left(\frac{4\pi}{\cancel{6}}\right) = \boxed{\frac{4\pi}{\sqrt{3}}}$$

Substitute $\boxed{\begin{aligned} u &= x+1 \\ du &= dx \end{aligned}}$ $\quad\boxed{\begin{aligned} x = s &\Rightarrow u = s+1 \\ x = 0 &\Rightarrow u = 1 \end{aligned}}$

9. $\displaystyle\int_3^\infty \frac{7}{x^2+3x-10}\,dx = \lim_{t\to\infty}\int_3^t \frac{7}{x^2+3x-10}\,dx = \lim_{t\to\infty}\int_3^t \frac{7}{(x-2)(x+5)}\,dx$

$$\overset{\text{PFD}}{=} \lim_{t\to\infty}\int_3^t \frac{1}{x-2}-\frac{1}{x+5}\,dx = \lim_{t\to\infty} \ln|x-2|-\ln|x+5|\Big|_3^t$$

$$= \lim_{t\to\infty} \ln|t-2|-\ln|t+5|-(\cancelto{0}{\ln 1}-\ln 8)$$

$$= \lim_{t\to\infty} \cancelto{\infty}{\ln|t-2|}-\cancelto{\infty}{\ln|t+5|}+\ln 8$$

$$= \ln\left|\lim_{t\to\infty}\frac{t-2}{t+5}^{\frac{\infty}{\infty}}\right|+\ln 8 \overset{\text{L'H}}{=} \ln\left|\lim_{t\to\infty}\cancelto{1}{\frac{1}{1}}\right|+\ln 8$$

$$= \cancelto{0}{\ln 1}+\ln 8 = 0+\ln 8 = \boxed{\ln 8}$$

Partial Fractions Decomposition:

$$\frac{7}{(x-2)(x+5)} = \frac{A}{x-2}+\frac{B}{x+5}$$

Clearing the denominator yields:

$7 = A(x+5)+B(x-2)$
$7 = (A+B)x+5A-2B$
so that $A+B=0$, and $5A-2B=7$
Solve for $A=1$, and $B=-1$

Note: you can also finish the last indeterminate limit using algebra

$$\ldots = \ln\left|\lim_{t\to\infty}\frac{t-2}{t+5}\cdot\frac{\frac{1}{t}}{\frac{1}{t}}\right|+\ln 8 = \ln\left|\lim_{t\to\infty}\frac{1-\cancelto{0}{\frac{2}{t}}}{1+\cancelto{0}{\frac{5}{t}}}\right|+\ln 8 = \cancelto{0}{\ln 1}+\ln 8 = \boxed{\ln 8}$$

$$
\begin{aligned}
10.\quad \int_e^\infty \frac{\ln x}{x^2}\,dx &= \lim_{t\to\infty}\int_e^t \ln x \cdot x^{-2}\,dx \\
&= \lim_{t\to\infty} -\frac{\ln x}{x}\bigg|_e^t + \int_e^t x^{-2}\,dx \\
&= \lim_{t\to\infty} -\frac{\ln x}{x}\bigg|_e^t - \frac{1}{x}\bigg|_e^t \\
&= \lim_{t\to\infty} -\frac{\ln t}{t} + \frac{\cancelto{1}{\ln e}}{e} - \frac{1}{t} + \frac{1}{e} \\
&= \lim_{t\to\infty} -\frac{\ln t}{t}^{\frac{\infty}{\infty}} + \frac{1}{e} - \cancelto{0}{\frac{1}{\cancelto{\infty}{t}}} + \frac{1}{e} \\
&\overset{\text{L'H}}{=} \lim_{t\to\infty} -\frac{\cancelto{0}{\frac{1}{t}}}{1} + \frac{1}{e} - \cancelto{0}{\frac{1}{t}} + \frac{1}{e} \\
&= 0 + \frac{1}{e} + 0 + \frac{1}{e} = \boxed{\frac{2}{e}}
\end{aligned}
$$

IBP: $\boxed{\begin{array}{ll} u = \ln x & dv = x^{-2}dx \\ du = \frac{1}{x}dx & v = -\frac{1}{x} \end{array}}$