Math 121 Self-Assessment Practice Worksheet #4 Answer Key

Compute each of the following Integrals.

1. $$\int x \arcsin x \, dx = \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\, dx$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{\sin^2\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos\theta} \, d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \sin^2\theta d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{2}\int \frac{1-\cos(2\theta)}{2} d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\int 1-\cos(2\theta) d\theta$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\left[\theta - \frac{\sin(2\theta)}{2}\right] + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta) + C$$

$$= \frac{x^2}{2}\arcsin x - \frac{1}{4}\theta + \frac{1}{8}\cdot 2\sin\theta\cos\theta + C$$

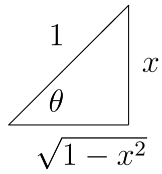
$$= \boxed{\frac{x^2}{2}\arcsin x - \frac{1}{4}\arcsin x + \frac{1}{4}x\sqrt{1-x^2} + C}$$

I.B.P.

$u = \arcsin x$	$dv = x dx$
$du = \frac{1}{\sqrt{1-x^2}} dx$	$v = \frac{x^2}{2}$

Trig Sub

$x = \sin\theta$
$dx = \cos\theta d\theta$



$$2.\ \int_0^{\frac{\pi}{2}} \frac{\cos x}{\left[1+\sin^2 x\right]^{\frac{7}{2}}}\ dx = \int_0^1 \frac{1}{[1+u^2]^{\frac{7}{2}}}\ du = \int_{u=0}^{u=1} \frac{1}{[1+\tan^2\theta]^{\frac{7}{2}}}\ \sec^2\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \frac{1}{[\sec^2\theta]^{\frac{7}{2}}}\ \sec^2\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \frac{1}{\left(\sqrt{\sec^2\theta}\right)^7}\ \sec^2\theta\ d\theta = \int_{u=0}^{u=1} \frac{1}{\sec^7\theta}\ \sec^2\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \frac{1}{\sec^5\theta}\ d\theta = \int_{u=0}^{u=1} \cos^5\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \cos^4\theta\cos\theta\ d\theta = \int_{u=0}^{u=1} \left(\cos^2\theta\right)^2\cos\theta\ d\theta$$

$$= \int_{u=0}^{u=1} \left(1-\sin^2\theta\right)^2\cos\theta\ d\theta$$

$$= \int_{u=0}^{u=1} (1-w^2)^2\ dw = \int_{u=0}^{u=1} 1-2w^2+w^4\ dw$$

$$= w - \frac{2w^3}{3} + \frac{w^5}{5}\Big|_{u=0}^{u=1} = \sin\theta - \frac{2\sin^3\theta}{3} + \frac{\sin^5\theta}{5}\Big|_{u=0}^{u=1}$$

$$= \frac{u}{\sqrt{u^2+1}} - \frac{2}{3}\left(\frac{u}{\sqrt{u^2+1}}\right)^3 + \frac{1}{5}\left(\frac{u}{\sqrt{u^2+1}}\right)^5\Big|_{u=0}^{u=1}$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5 - (0-0+0)$$

$$= \frac{1}{\sqrt{2}} - \frac{2}{3}\left(\frac{1}{2\sqrt{2}}\right) + \frac{1}{5}\left(\frac{1}{4\sqrt{2}}\right) = \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}}$$

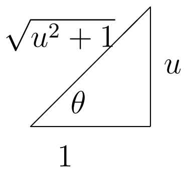
$$= \frac{60}{60\sqrt{2}} - \frac{20}{60\sqrt{2}} + \frac{3}{60\sqrt{2}} = \boxed{\frac{43}{60\sqrt{2}}}$$

Standard u substitution to simplify at the start:

$u = \sin x$	$x = 0 \Rightarrow u = 0$
$du = \cos x dx$	$x = \frac{\pi}{2} \Rightarrow u = 1$

Trig. Substitute

$$\boxed{\begin{array}{l} u = \tan\theta \\ \\ du = \sec^2\theta d\theta \end{array}}$$



Standard w substitution for *odd* trig. integral $\int \cos^5\theta \; d\theta$ technique:

$$\boxed{\begin{array}{l} w = \sin\theta \\ \\ dw = \cos\theta \; d\theta \end{array}}$$

you can also change one more limit if you rather

$$\ldots = \int_0^{\frac{1}{\sqrt{2}}} (1-w^2)^2 \; dw = \ldots$$

3. $$\int \frac{x^2}{\sqrt{4-x^2}} \; dx = \int \frac{(2\sin\theta)^2}{\sqrt{4-4\sin^2\theta}} \; 2\cos\theta \; d\theta = \int \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}} \; 2\cos\theta \; d\theta$$

$$= 4\int \frac{\sin^2\theta}{\sqrt{4}\sqrt{\cos^2\theta}} \; 2\cos\theta \; d\theta = 4\int \frac{\sin^2\theta}{\cancel{2\cos\theta}} \; \cancel{2\cos\theta} \; d\theta$$

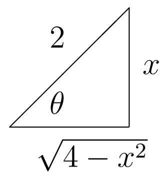
$$= 4\int \sin^2\theta \; d\theta = 4\int \frac{1-\cos(2\theta)}{2} \; d\theta = \frac{4}{2}\int 1-\cos(2\theta) \; d\theta$$

$$= 2\left(\theta - \frac{\sin(2\theta)}{2}\right) + C = 2\left(\theta - \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C$$

$$= \boxed{2\left(\arcsin\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\right) + C}$$

Trig. Substitute

$$\boxed{\begin{array}{l} x = 2\sin\theta \Rightarrow \theta = \arcsin\left(\frac{x}{2}\right) \\ \\ dx = 2\cos\theta d\theta \end{array}}$$



4. $$\int x^5\sqrt{1-x^2}\,dx = \int \sin^5\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$

$$= \int \sin^5\theta\cos\theta\cos\theta\,d\theta = \int \sin^5\theta\cos^2\theta\,d\theta$$

$$= \int \sin^4\theta\cos^2\theta\ \sin\theta\,d\theta = \int (\sin^2\theta)^2\cos^2\theta\ \sin\theta\,d\theta$$

$$= \int (1-\cos^2\theta)^2\cos^2\theta\ \sin\theta\,d\theta$$

$$= -\int (1-w^2)^2 w^2\,dw$$

$$= -\int (1-2w^2+w^4)w^2\,dw$$

$$= -\int w^2-2w^4+w^6\,dw$$

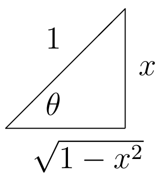
$$= -\left(\frac{w^3}{3}-\frac{2w^5}{5}+\frac{w^7}{7}\right)+C$$

$$= -\left(\frac{\cos^3\theta}{3}-\frac{2\cos^5\theta}{5}+\frac{\cos^7\theta}{7}\right)+C$$

$$= \boxed{-\left(\frac{(\sqrt{1-x^2})^3}{3}-\frac{2(\sqrt{1-x^2})^5}{5}+\frac{(\sqrt{1-x^2})^7}{7}\right)+C}$$

$$\overset{\text{OR}}{=} -\frac{(1-x^2)^{\frac{3}{2}}}{3}+\frac{2(1-x^2)^{\frac{5}{2}}}{5}-\frac{(1-x^2)^{\frac{7}{2}}}{7}+C$$

Trig. Substitute $\boxed{\begin{array}{l} x=\sin\theta \\ \\ dx=\cos\theta d\theta \end{array}}$



Substitute $\boxed{\begin{array}{l} w=\cos\theta \\ dw=-\sin\theta d\theta \\ -dw=\sin\theta d\theta \end{array}}$

Note: You can also do an *inverted/reversed* u-substitution, but the problem required Trig Sub here.

$$5.\ \int_{-3}^{3} \sqrt{9-x^2}\ dx = \int_{x=-3}^{x=3} \sqrt{9-9\sin^2\theta}\ 3\cos\theta\ d\theta = \int_{x=-3}^{x=3} \sqrt{9\left(1-\sin^2\theta\right)}\ 3\cos\theta\ d\theta$$

$$= \int_{x=-3}^{x=3} \sqrt{9\cos^2\theta}\ 3\cos\theta\ d\theta = \int_{x=-3}^{x=3} 3\cos\theta \cdot\ 3\cos\theta\ d\theta$$

$$= 9\int_{x=-3}^{x=3} \cos^2\theta\ d\theta = 9\int_{x=-3}^{x=3} \frac{1+\cos(2\theta)}{2}\ d\theta$$

$$= \frac{9}{2}\int_{x=-3}^{x=3} 1+\cos(2\theta)\ d\theta = \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Bigg|_{x=-3}^{x=3}$$

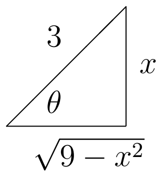
$$= \frac{9}{2}\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right)\Bigg|_{x=-3}^{x=3}$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right)\Bigg|_{-3}^{3}$$

$$= \frac{9}{2}\left[(\arcsin 1 + (1)(0)) - (\arcsin(-1) + (-1)(0))\right]$$

$$= \frac{9}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{9}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \boxed{\frac{9\pi}{2}}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta d\theta \end{array}}$



Note: if you want to *change* your Limits of Integration instead...

$$\int_{-3}^{3} \sqrt{9-x^2}\ dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\ 3\cos\theta\ d\theta \ldots = \frac{9}{2}\left(\theta + \sin\theta\cos\theta\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{9}{2}\left[\left(\frac{\pi}{2} + \sin\left(\frac{\pi}{2}\right)\cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\right) - \left(-\frac{\pi}{2} + \sin\left(-\frac{\pi}{2}\right)\cancelto{0}{\cos\left(-\frac{\pi}{2}\right)}\right)\right]$$

$$= \frac{9\pi}{2}$$

$$6.\ \int x^4 \arcsin x\ dx = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{x^5}{\sqrt{1-x^2}}\ dx$$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\sqrt{1-\sin^2\theta}}\ \cos\theta\ d\theta$$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\sqrt{\cos^2\theta}}\ \cos\theta\ d\theta$$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \frac{\sin^5\theta}{\cancel{\cos\theta}}\ \cancel{\cos\theta}\ d\theta$$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^5\theta\ d\theta = \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \sin^4\theta\sin\theta\ d\theta$$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int \left(\sin^2\theta\right)^2\sin\theta\ d\theta$$

$$= \frac{x^5}{5}\arcsin x - \frac{1}{5}\int (1-\cos^2\theta)^2\sin\theta\ d\theta$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int (1-w)^2\ dw$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\int 1-2w^2+w^4\ dw$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left(w - \frac{2w^3}{3} + \frac{w^5}{5}\right) + C$$

$$= \frac{x^5}{5}\arcsin x + \frac{1}{5}\left(\cos\theta - \frac{2\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right) + C$$

$$= \boxed{\frac{x^5}{5}\arcsin x + \frac{\sqrt{1-x^2}}{5} - \frac{2\left(\sqrt{1-x^2}\right)^3}{15} + \frac{\left(\sqrt{1-x^2}\right)^5}{25} + C}$$

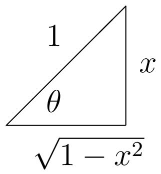
$u = \arcsin x \qquad dv = x^4 dx$

$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = \frac{x^5}{5}$

Trig. Sub

$x = \sin\theta$

$dx = \cos\theta d\theta$



Sub

$w = \cos\theta$

$dw = -\sin\theta d\theta$

$-dw = \sin\theta d\theta$

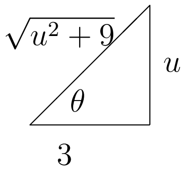
7. $\displaystyle\int \frac{1}{x\left(9+(\ln x)^2\right)}\,dx = \int \frac{1}{9+u^2}\,du = \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \boxed{\frac{1}{3}\arctan\left(\frac{\ln x}{3}\right) + C}$

Substitute $\boxed{\begin{array}{l} u = \ln x \\ du = \dfrac{1}{x}dx \end{array}}$

8. $$\begin{aligned}
\int \frac{1}{x\left(9+(\ln x)^2\right)^{\frac{5}{2}}}\,dx &= \int \frac{1}{\left(9+u^2\right)^{\frac{5}{2}}}\,du = \int \frac{1}{\left(\sqrt{9+u^2}\right)^5}\,du \\
&= \int \frac{1}{\left(\sqrt{9+9\tan^2\theta}\right)^5}\,3\sec^2\theta\,d\theta \\
&= \int \frac{1}{\left(\sqrt{9(1+\tan^2\theta)}\right)^5}\,3\sec^2\theta\,d\theta \\
&= \int \frac{1}{\left(3\sqrt{\sec^2\theta}\right)^5}\,3\sec^2\theta\,d\theta \\
&= \int \frac{1}{(3\sec\theta)^5}\,3\sec^2\theta\,d\theta = \frac{3}{3^5}\int \frac{1}{\sec^5\theta}\,\sec^2\theta\,d\theta \\
&= \frac{1}{3^4}\int \frac{1}{\sec^3\theta}\,d\theta = \frac{1}{81}\int \cos^3\theta\,d\theta \\
&= \frac{1}{81}\int \cos^2\theta\cos\theta\,d\theta = \frac{1}{81}\int \left(1-\sin^2\theta\right)\cos\theta\,d\theta \\
&= \frac{1}{81}\int \left(1-w^2\right)\,dw = \frac{1}{81}\left(w - \frac{w^3}{3}\right) + C \\
&= \frac{1}{81}\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + C \\
&= \frac{1}{81}\left(\frac{u}{\sqrt{u^2+9}} - \frac{1}{3}\left(\frac{u}{\sqrt{u^2+9}}\right)^3\right) + C \\
&= \boxed{\frac{1}{81}\left(\frac{\ln x}{\sqrt{(\ln x)^2+9}} - \frac{1}{3}\left(\frac{\ln x}{\sqrt{(\ln x)^2+9}}\right)^3\right) + C}
\end{aligned}$$

Substitute $\boxed{\begin{array}{l} u = \ln x \\ du = \dfrac{1}{x} dx \end{array}}$ Substitute $\boxed{\begin{array}{l} w = \sin\theta \\ dw = \cos\theta d\theta \end{array}}$

Trig. Substitute $\boxed{\begin{array}{l} u = 3\tan\theta \\ \\ du = 3\sec^2\theta d\theta \end{array}}$



$$9. \int \frac{1}{(x^2+4)^2}\,dx = \int \frac{1}{(4\tan^2\theta+4)^2}\,2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{(4\sec^2\theta)^2}\,2\sec^2\theta\,d\theta = \int \frac{1}{16\sec^4\theta}\,2\sec^2\theta\,d\theta$$

$$= \frac{1}{8}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{1}{8}\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\,d\theta = \frac{1}{8}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

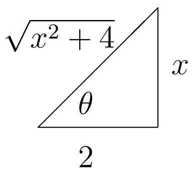
$$= \frac{1}{16}\int 1+\cos(2\theta)\,d\theta = \frac{1}{16}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C$$

$$= \frac{1}{16}\left(\theta + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}\right) + C = \frac{1}{16}\left(\theta + \sin\theta\cos\theta\right) + C$$

$$= \frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \left(\frac{x}{\sqrt{x^2+4}}\right)\left(\frac{2}{\sqrt{x^2+4}}\right)\right) + C$$

$$= \boxed{\frac{1}{16}\left(\arctan\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right) + C}$$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\tan\theta \\ \\ dx = 2\sec^2\theta d\theta \end{array}}$



10. $$\int \frac{x}{(4+x^2)^{\frac{7}{2}}}\ dx = \int \frac{x}{\left(\sqrt{4+x^2}\right)^7}\ dx$$

$$= \int \frac{2\tan\theta}{\left(\sqrt{4+4\tan^2\theta}\right)^7}\ 2\sec^2\theta\ d\theta$$

$$= \int \frac{2\tan\theta}{\left(\sqrt{4(1+\tan^2\theta)}\right)^7}\ 2\sec^2\theta\ d\theta$$

$$= \int \frac{2\tan\theta}{\left(\sqrt{4(\sec^2\theta)}\right)^7}\ 2\sec^2\theta\ d\theta$$

$$= \int \frac{2\tan\theta}{(2\sec\theta)^7}\ 2\sec^2\theta\ d\theta$$

$$= \frac{2^2}{2^7}\int \frac{\tan\theta}{\sec^7\theta}\ \sec^2\theta\ d\theta$$

$$= \frac{1}{2^5}\int \frac{\tan\theta}{\sec^5\theta}\ d\theta = \frac{1}{32}\int \tan\theta\cdot\cos^5\theta\ d\theta$$

$$= \frac{1}{32}\int \frac{\sin\theta}{\cos\theta}\cdot\cos^5\theta\ d\theta$$

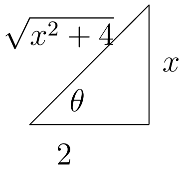
$$= \frac{1}{32}\int \sin\theta\cdot\cos^4\theta\ d\theta \quad \text{Classic } u\text{-substitution}$$

$$= -\frac{1}{32}\int w^4\ dw = -\frac{1}{32}\left(\frac{w^5}{5}\right) + C$$

$$= -\frac{1}{160}\cos^5\theta + C = \boxed{-\frac{1}{160}\left(\frac{2}{\sqrt{x^2+4}}\right)^5 + C}$$

$$\overset{\text{OR}}{=} -\frac{1}{32}\cdot\frac{1}{5}\left(\frac{32}{(x^2+4)^{\frac{5}{2}}}\right) + C = -\frac{1}{5\,(x^2+4)^{\frac{5}{2}}} + C$$

Trig. Substitute $\boxed{\begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta d\theta \end{array}}$



Sub $\boxed{\begin{array}{l} w = \cos\theta \\ dw = -\sin\theta\ d\theta \\ -dw = \sin\theta\ d\theta \end{array}}$