Math 121 Self-Assessment Practice Worksheet #2 Answer Key

1. $\displaystyle\int_3^{3\sqrt{3}} \frac{1}{\sqrt{36-x^2}} + \frac{1}{9+x^2}\ dx \overset{a\text{-rule}}{=} \arcsin\left(\frac{x}{6}\right)\Bigg|_3^{3\sqrt{3}} + \frac{1}{3}\arctan\left(\frac{x}{3}\right)\Bigg|_3^{3\sqrt{3}}$

$$= \arcsin\left(\frac{3\sqrt{3}}{6}\right) - \arcsin\left(\frac{3}{6}\right) + \frac{1}{3}\arctan\left(\frac{3\sqrt{3}}{3}\right) - \frac{1}{3}\arctan\left(\frac{3}{3}\right)$$

$$= \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right) + \frac{1}{3}\arctan\left(\sqrt{3}\right) - \frac{1}{3}\arctan(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{3}\right) - \frac{1}{3}\left(\frac{\pi}{4}\right) = \frac{\pi}{3} - \frac{\pi}{6} + \frac{\pi}{9} - \frac{\pi}{12} = \frac{12\pi}{36} - \frac{6\pi}{36} + \frac{4\pi}{36} - \frac{3\pi}{36} = \boxed{\frac{7\pi}{36}}$$

2. $\displaystyle\int \frac{4}{(1+x^2)(1+(\arctan x)^2)}\ dx = 4\int \frac{1}{1+w^2}\ dw$

$$= 4\arctan w + C = \boxed{4\arctan(\arctan x) + C}$$

$$\boxed{\begin{aligned} w &= \arctan x \\ dw &= \frac{1}{1+x^2}\ dx \end{aligned}}$$

3. $\displaystyle\int_0^{\ln\sqrt{2}} \frac{e^x}{\sqrt{4-e^{2x}}}\ dx = \int_0^{\ln\sqrt{2}} \frac{e^x}{\sqrt{4-(e^x)^2}}\ dx = \int_1^{\sqrt{2}} \frac{1}{\sqrt{4-u^2}}\ du$

$$\overset{a\text{-rule}}{=} \arcsin\left(\frac{u}{2}\right)\Bigg|_1^{\sqrt{2}} = \arcsin\left(\frac{\sqrt{2}}{2}\right) - \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{4} - \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12} = \boxed{\frac{\pi}{12}}$$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x\ dx \end{aligned}} \qquad \boxed{\begin{aligned} &x = 0 \Rightarrow u = e^0 = 1 \\ &x = \ln\sqrt{2} \Rightarrow u = e^{\ln\sqrt{2}} = \sqrt{2} \end{aligned}}$$

4. $\displaystyle\int_e^{e^3} \frac{1}{x\,(3+(\ln x)^2)}\ dx = \int_1^3 \frac{1}{3+u^2}\ du$

$$\overset{a\text{-rule}}{=} \frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)\Bigg|_1^3 = \frac{1}{\sqrt{3}}\left(\arctan\left(\cancelto{\sqrt{3}}{\frac{3}{\sqrt{3}}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right)$$

$$= \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{1}{\sqrt{3}} \left(\frac{2\pi}{6} - \frac{\pi}{6} \right) = \frac{1}{\sqrt{3}} \left(\frac{\pi}{6} \right) = \boxed{\frac{\pi}{6\sqrt{3}}}$$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\ dx \end{aligned}} \quad \boxed{\begin{aligned} x = e &\Rightarrow u = \ln e = 1 \\ x = e^3 &\Rightarrow u = \ln e^3 = 3 \end{aligned}}$$

5. $$\int_{\frac{\pi}{2}}^{\pi} \frac{\cos x}{3 + \sin^2 x}\ dx = \int_1^0 \frac{1}{3 + u^2} \overset{a\text{-rule}}{=} \frac{1}{\sqrt{3}} \arctan \left(\frac{u}{\sqrt{3}} \right) \Big|_1^0$$

$$= \frac{1}{\sqrt{3}} \left(\arctan 0 - \arctan \left(\frac{1}{\sqrt{3}} \right) \right) = \frac{1}{\sqrt{3}} \left(0 - \frac{\pi}{6} \right) = \boxed{-\frac{\pi}{6\sqrt{3}}}$$

$$\boxed{\begin{aligned} u &= \sin x \\ du &= \cos x\ dx \end{aligned}} \quad \boxed{\begin{aligned} x = \tfrac{\pi}{2} &\Rightarrow u = \sin \tfrac{\pi}{2} = 1 \\ x = \pi &\Rightarrow u = \sin \pi = 0 \end{aligned}}$$

6. $$\int_3^9 \frac{1}{\sqrt{x}(9 + x)}\ dx = \int_3^9 \frac{1}{\sqrt{x}(9 + (\sqrt{x})^2)}\ dx = 2 \int_{\sqrt{3}}^3 \frac{1}{9 + u^2}\ du$$

$$\overset{a\text{-rule}}{=} 2 \left(\frac{1}{3} \right) \arctan \left(\frac{u}{3} \right) \Big|_{\sqrt{3}}^3 = \frac{2}{3} \left(\arctan 1 - \arctan \left(\cancelto{\frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{3}} \right) \right)$$

$$= \frac{2}{3} \left(\frac{\pi}{4} - \frac{\pi}{6} \right) = \frac{2}{3} \left(\frac{3\pi}{12} - \frac{2\pi}{12} \right) = \frac{2}{3} \left(\frac{\pi}{12} \right) = \boxed{\frac{\pi}{18}}$$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \tfrac{1}{2\sqrt{x}}\ dx \\ 2du &= \tfrac{1}{\sqrt{x}}\ dx \end{aligned}} \quad \boxed{\begin{aligned} x = 3 &\Rightarrow u = \sqrt{3} \\ x = 9 &\Rightarrow u = \sqrt{9} = 3 \end{aligned}}$$

7. $$\int \frac{x^2 + x + 1}{x^2 + 4}\ dx \overset{\text{split}}{=} \int \frac{x^2}{x^2 + 4} + \frac{x}{x^2 + 4} + \frac{1}{x^2 + 4}\ dx$$

$$\overset{\text{slip-in/slip-out}}{=} \int \frac{x^2 + 4 - 4}{x^2 + 4} + \frac{x}{x^2 + 4} + \frac{1}{x^2 + 4}\ dx$$

$$\overset{\text{split}}{=} \int \frac{\cancel{x^2+4}^{\,1}}{\cancel{x^2+4}} - \frac{4}{x^2+4} + \frac{x}{x^2+4} + \frac{1}{x^2+4}\, dx \quad \text{can also combine common factors earlier}$$

$$\overset{*}{=} x - \frac{\cancel{4}^{\,2}}{\cancel{2}} \arctan\left(\frac{x}{2}\right) + \frac{1}{2}\int \frac{1}{w}\, dw + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$= x - 2\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln|w| + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$= \boxed{x - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln|x^2+4| + C}$$

* middle u-sub piece

$$\boxed{\begin{array}{l} w = x^2 + 4 \\ du = 2x\ dx \\ \frac{1}{2}du = x\ dx \end{array}}$$

Note: You can also adjust the algebra at the start to simplify a few steps.

$$\int \frac{x^2+x+1}{x^2+4}\, dx = \int \frac{x^2+4+x-3}{x^2+4}\, dx = \int \frac{x^2+4}{x^2+4}\, dx + \int \frac{x}{x^2+4}\, dx - \int \frac{3}{x^2+4}\, dx = \ldots$$

8. (a) Use implicit differentiation to **PROVE** that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

If $y = \arctan x$, then Invert to get $\tan y = x$.

Implicitly differentiate both sides,

$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$.

Then $\sec^2 y\, \frac{dy}{dx} = 1$.

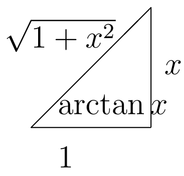
Solve for $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$

Here we used the trig. identity to finish

OR finish

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2(\arctan x)} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

using the trig. from the triangle



(b) From part (a) we now know that $\frac{1}{1+x^2}\,dx = \arctan x + C$. Use this fact **and integration** to **PROVE** that $\int \frac{1}{3+x^2}\,dx = \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$

$$\int \frac{1}{3+x^2}\,dx = \int \frac{1}{3\left(1+\frac{x^2}{3}\right)}\,dx = \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{3}\int \frac{1}{1+u^2}\,du = \frac{1}{\sqrt{3}}\arctan u + C = \frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C$$

Standard u substitution to simplify:

$$u = \frac{x}{\sqrt{3}}$$

$$du = \frac{1}{\sqrt{3}}\,dx$$

$$\sqrt{3}du = \ dx$$

Note: **OR** you can also do a trig. substitution here (in the future).

9. Use implicit differentiation to **PROVE** that $\frac{d}{dx}\sin^{-1}(5x) = \frac{5}{\sqrt{1-25x^2}}$

Let $y = \sin^{-1}(5x)$. Then Invert to get $\sin y = 5x$. Implicitly differentiate both sides, $\frac{d}{dx}(\sin y) = \frac{d}{dx}(5x)$.

Then $\cos y\,\frac{dy}{dx} = 5$ Solve for $\frac{dy}{dx} = \frac{5}{\cos y} = \frac{5}{\sqrt{1-\sin^2 y}} = \frac{5}{\sqrt{1-25x^2}}$