Math 121 Self-Assessment Practice Worksheet #11 Answer Key

Find the **Sum** for each of the following series.

1.

$$\begin{aligned} 1-2+\frac{4}{2!}-\frac{8}{3!}+\frac{16}{4!}-\frac{32}{5!}+\dots &= 1-2+\frac{2^2}{2!}-\frac{2^3}{3!}+\frac{2^4}{4!}-\frac{2^5}{5!}+\dots \\ &= 1+(-2)+\frac{(-2)^2}{2!}+\frac{(-2)^3}{3!}+\frac{(-2)^4}{4!}+\frac{(-2)^5}{5!}+\dots \\ &= \boxed{e^{-2}} \end{aligned}$$

Recall: $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$

2.

$$\begin{aligned} \frac{1}{3\pi}-\frac{1}{18\pi^2}+\frac{1}{81\pi^3}-\frac{1}{324\pi^4}+\dots &= \frac{1}{3\pi}-\frac{1}{2\cdot 9\pi^2}+\frac{1}{3\cdot(27)\pi^3}-\frac{1}{4\cdot(81)\pi^4}+\dots \\ &= \frac{1}{3\pi}-\frac{1}{2(3\pi)^2}+\frac{1}{3(3\pi)^3}-\frac{1}{4(3\pi)^4}+\dots \\ &= \boxed{\ln\left(1+\frac{1}{3\pi}\right)} \end{aligned}$$

Recall: $\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots$

3.

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,(\ln 9)^n}{2^{n+1}\,n!} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-\ln 9)^n}{2^n\,n!} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{\ln 9}{2}\right)^n}{n!} = \frac{1}{2}e^{\left(-\frac{\ln 9}{2}\right)}$$

$$= \frac{1}{2}\cancel{e}^{\cancel{\ln}\left(9^{-\frac{1}{2}}\right)} = \frac{1}{2}\left(9^{-\frac{1}{2}}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{9}}\right) = \frac{1}{2}\left(\frac{1}{3}\right) = \boxed{\frac{1}{6}}$$

Recall: $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and that the series for e^x is not alternating. Absorb the minus.

4. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \arctan(1) = \boxed{\frac{\pi}{4}}$

Recall: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

5. $-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots = \left(\cancelto{-1}{\cos \pi}\right) - 1 = -1 - 1 = \boxed{-2}$

Recall: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

Note: this is the formula for $\cos(\pi)$ but we're missing the first term of 1 so we subtract the 1 from the *full* sum $\cos 1 = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \frac{\pi^8}{8!} - \ldots$

6. $$\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{9^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{3^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{3}\right)^{2n}}{(2n+1)!} \cdot \frac{\left(\frac{\pi}{3}\right)}{\left(\frac{\pi}{3}\right)}$$

$$= \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!} = \frac{3}{\pi} \sin\left(\frac{\pi}{3}\right) = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{2\pi}}$$

Recall: $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

7. $$\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{2^{4n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(2^2)^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{4^{2n}\ (2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{4}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$$

Recall: $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

8. $\frac{1}{6} - \frac{1}{2\cdot 6^2} + \frac{1}{3\cdot 6^3} - \frac{1}{4\cdot 6^4} + \ldots = \frac{1}{6} - \frac{\left(\frac{1}{6}\right)^2}{2} + \frac{\left(\frac{1}{6}\right)^3}{3} - \frac{\left(\frac{1}{6}\right)^4}{4} + \ldots = \ln\left(1+\frac{1}{6}\right) = \boxed{\ln\left(\frac{7}{6}\right)}$

Recall: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

9. $\sum_{n=0}^{\infty} \frac{1}{3!\pi^n} = \frac{1}{3!}\sum_{n=0}^{\infty}\left(\frac{1}{\pi}\right)^n = \frac{1}{6}\left(\frac{1}{1-\frac{1}{\pi}}\right) = \frac{1}{6}\left(\frac{1}{\frac{\pi-1}{\pi}}\right) = \frac{1}{6}\left(\frac{\pi}{\pi-1}\right) = \boxed{\frac{\pi}{6(\pi-1)}}$

Recall: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

10. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\ \pi^{2n+1}}{9\ (2n)!} = -\frac{\pi}{9}\sum_{n=0}^{\infty}\frac{(-1)^n\ \pi^{2n}}{(2n)!} = -\frac{\pi}{9}\cos\pi = -\frac{\pi}{9}(-1) = \boxed{\frac{\pi}{9}}$

Recall: $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

11. $\pi^2 - \frac{\pi^4}{3!} + \frac{\pi^6}{5!} - \frac{\pi^8}{7!} + \ldots = \pi\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots\right) = \pi\cancelto{0}{\sin\pi} = \boxed{0}$

Recall: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

12. $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\pi^{2n}}{4^n(2n+1)!} = -\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{2^{2n}(2n+1)!} = -\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{2}\right)^{2n}}{(2n+1)!}\cdot\frac{\left(\frac{\pi}{2}\right)}{\left(\frac{\pi}{2}\right)}$

$$= -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!} = -\frac{2}{\pi}\cancelto{1}{\sin\left(\frac{\pi}{2}\right)} = \boxed{-\frac{2}{\pi}}$$

Recall: $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$

13. $$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1} (\ln 4)^n}{n!} = 3\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (\ln 4)^n}{n!} = 3\sum_{n=0}^{\infty} \frac{(-3\ln 4)^n}{n!}$$

$$= 3e^{-3\ln 4} = 3\cancel{e}^{\cancel{\ln}\left(4^{-3}\right)} = 3\left(4^{-3}\right) = \frac{3}{4^3} = \boxed{\frac{3}{64}}$$

Recall: $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and that the series for e^x is not alternating. Absorb the minus.

14. $$1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \ldots = 1 + (-e) + \frac{(-e)^2}{2!} + \frac{(-e)^3}{3!} + \frac{(-e)^4}{4!} + \ldots = \boxed{e^{-e}}$$

Recall: $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

15. $$-\pi + \frac{\pi^3}{3!} - \frac{\pi^5}{5!} + \frac{\pi^7}{7!} - \ldots = -\left(\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \ldots\right) = \cancel{-\sin\pi} \overset{0}{=} \boxed{0}$$

Recall: $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

16. $$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \ldots = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\right) = -\ln(1+1) = \boxed{-\ln 2}$$

Recall: $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

17. $$-\frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} + \ldots = 2\left(-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\right)$$

$$= 2\left[\ln(1+1) - 1\right] = 2\left[\ln 2 - 1\right] \overset{\text{or}}{=} \boxed{2\ln 2 - 2}$$

Recall: $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

Recall: $\ln(2) = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots$ so we have the *full* sum minus the first typical term of 1

18. $1+1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\frac{\pi^8}{8!}-\ldots = 1+\cancelto{-1}{\cos\pi} = 1-1 = \boxed{0}$

Recall: $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

19. $\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots = \frac{(-1)^2}{2!}+\frac{(-1)^3}{3!}+\frac{(-1)^4}{4!}+\frac{(-1)^5}{5!}+\ldots = e^{-1}$

Recall: $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$

Note: $e^{-1} = \cancelto{0}{1+(-1)}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots$ and by chance the first two terms $1-1$ end up cancelling giving the original sum.

20.

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\;3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\;(\sqrt{3})^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\;(\sqrt{3})^{2n}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\;(\sqrt{3})^{2n+1}} = \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$= \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) = \sqrt{3}\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}\pi}{6}}$$

Recall: $\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$

21. Find the MacLaurin Series for $\ln(1+6x)$.

Hint: $\ln(1+6x) = \int \cancelto{\text{P.S.}}{\frac{6}{1+6x}}\,dx$. Yes, solve for $+C$.

First derive it using substitution and integration.

$$\begin{aligned}\ln(1+6x) &= \int \frac{6}{1+6x}\,dx = \int \frac{6}{1-(-6x)}\,dx \\ &= 6\int \sum_{n=0}^{\infty}(-6x)^n\,dx = 6\int \sum_{n=0}^{\infty}(-1)^n 6^n x^n\,dx \\ &= \int \sum_{n=0}^{\infty}(-1)^n 6^{n+1} x^n\,dx \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n 6^{n+1} x^{n+1}}{n+1} + \cancelto{0}{C} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 6^{n+1} x^{n+1}}{n+1}}\end{aligned}$$

To solve for $+C$, first expand this equation

$$\ln(1+6x) = 6x - \frac{6^2x^2}{2} + \frac{6^3x^3}{3} - \frac{6^4x^4}{4} + \frac{6^5x^5}{5} - \ldots + C$$

Test $x = 0$ into both sides of the equation above.

Note that $x = 0$ is in the Interval of Convergence for this series because it is the *Center* point of this power series.

$$\cancelto{0}{\ln 1} = 0 - \frac{0}{2} + \frac{0}{3} - \frac{0}{4} + \frac{0}{5} - \ldots + C$$

That is, $0 = 0 - 0 + 0 - 0 + 0 - \ldots + C \Rightarrow C = 0$, Substitute above.

Finally, $\ln(1+6x) = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 6^{n+1} x^{n+1}}{n+1}}$

22. Find the MacLaurin Series for $\ln(9+x^2)$.

Hint: $\ln(9+x^2) = \int \cancelto{\text{P.S.}}{\frac{2x}{9+x^2}}\,dx$. Yes, solve for $+C$.

First derive it using substitution and integration.

$$\ln(9+x^2) = \int \frac{2x}{9+x^2}\,dx = \int 2x\left(\frac{1}{9+x^2}\right)\,dx = \int \frac{2x}{9}\left(\frac{1}{1+\frac{x^2}{9}}\right)\,dx$$

$$= \int \frac{2x}{9}\left(\frac{1}{1-\left(-\frac{x^2}{9}\right)}\right)\,dx = \int \frac{2x}{9}\sum_{n=0}^{\infty}\left(-\frac{x^2}{9}\right)^n\,dx$$

$$= \int \frac{2x}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}\,dx = 2\int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{9^{n+1}}\,dx$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)} + \overset{\ln 9}{\cancel{C}} = \boxed{2\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)} + \ln 9}$$

To solve for $+C$, first expand this equation

$$\ln(9+x^2) = 2\left(\frac{x^2}{9\cdot 2} - \frac{x^4}{9^2\cdot 4} + \frac{x^6}{9^3\cdot 6} - \frac{x^8}{9^4\cdot 8} + \ldots + C\right)$$

Test $x=0$ into both sides of the equation above.

Note that $x=0$ is in the Interval of Convergence for this series because it is the *Center* point of this power series.

$$\overset{\ln 9}{\cancel{\ln(9+0)}} = \frac{0}{9\cdot 2} - \frac{0}{9^2\cdot 4} + \frac{0}{9^3\cdot 6} - \frac{0}{9^4\cdot 8} + C$$

That is, $\ln 9 = 0 - 0 + 0 - 0 + 0 - \ldots + C \Rightarrow C = \ln 9$, Substitute above.

Finally, $\ln(9+x^2) = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+2}}{9^{n+1}(2n+2)} + \ln 9}$

23. Show that the MacLaurin Series for $\frac{1}{2}\left(e^x - e^{-x}\right) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\frac{1}{2}\left(e^x - e^{-x}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right)$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) - \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[\left(\cancel{1} + x + \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} + \frac{x^5}{5!} + \ldots\right) \cancel{-1} + x - \cancel{\frac{x^2}{2!}} + \frac{x^3}{3!} - \cancel{\frac{x^4}{4!}} + \frac{x^5}{5!} + \ldots\right]$$

$$= \frac{1}{2}\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}x + \frac{\cancel{2}x^3}{3!} + \frac{\cancel{2}x^5}{5!} + \ldots\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$$

24. Show that the MacLaurin Series for $\frac{1}{2}\left(e^x + e^{-x}\right) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$$\frac{1}{2}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right)$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\right) + \left(1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \frac{(-x)^5}{5!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[\left(1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} + \cancel{\frac{x^5}{5!}} + \ldots\right) + \left(1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} - \cancel{\frac{x^5}{5!}} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2} + \frac{\cancel{2}x^2}{2!} + \frac{\cancel{2}x^4}{4!} + \ldots\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

25. Use Series to Estimate $\displaystyle\int_0^1 x^3 \sin\left(x^2\right)\, dx$ with error less than $\dfrac{1}{100}$. Justify.

Recall: $\displaystyle\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\int_0^1 x^3 \sin\left(x^2\right)\, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \left(x^2\right)^{2n+1}}{(2n+1)!}\, dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\, dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+5}}{(2n+1)!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{(2n+1)!(4n+6)}\Bigg|_0^1$$

$$= \frac{x^6}{6} - \frac{x^{10}}{3!(10)} + \frac{x^{14}}{5!(14)} - \ldots \Bigg|_0^1$$

$$= \frac{1}{6} - \frac{1}{60} + \frac{1}{1680} + \ldots - (0 - 0 + 0 - \ldots)$$

$$\approx \frac{1}{6} - \frac{1}{60} = \frac{10}{60} - \frac{1}{60} = \frac{9}{60} = \boxed{\frac{3}{20}} \leftarrow \text{estimate}$$

Using the Alternating Series Estimation Theorem, if we approximate the actual sum with only the first two terms, the error from the actual sum will be *at most* the absolute value of the next (first neglected) term, $\dfrac{1}{1680}$. Here $\dfrac{1}{1680} < \dfrac{1}{100}$ as desired.

26. Use Series to Estimate $\displaystyle\int_0^1 x^4 e^{-x^3}\, dx$ with error less than $\dfrac{1}{10}$. Justify.

Recall $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$
\begin{aligned}
\int_0^1 x^4 e^{-x^3}\,dx &= \int_0^1 x^4 \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!}\,dx = \int_0^1 x^4 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}\,dx \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+4}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+5}}{n!(3n+5)}\Bigg|_0^1 \\
&= \frac{x^5}{0!(5)} - \frac{x^8}{1!(8)} + \frac{x^{11}}{2!(11)} - \ldots\Bigg|_0^1 \\
&= \frac{1}{5} - \frac{1}{8} + \frac{1}{22} - \ldots - (0 - 0 + 0 - 0 + \ldots) \\
&\approx \frac{1}{5} - \frac{1}{8} = \frac{8}{40} - \frac{5}{40} = \boxed{\frac{3}{40}} \quad \leftarrow \text{estimate}
\end{aligned}
$$

Note this is an alternating series. Use the Alternating Series Estimation Theorem (ASET). If we approximate the actual sum with only the first two terms, the error from the actual sum will be *at most* the absolute value of the next (first neglected) term, $\frac{1}{22}$. Here $\frac{1}{22} < \frac{1}{10}$ as desired.

27. Estimate $\frac{1}{\sqrt{e}}$ with error less than $\frac{1}{100}$. Justify.

Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$$
\begin{aligned}
\frac{1}{\sqrt{e}} &= e^{-\frac{1}{2}} = 1 - \frac{1}{2} + \frac{\left(-\frac{1}{2}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)^3}{3!} + \frac{\left(-\frac{1}{2}\right)^4}{4!} + \ldots \\
&= 1 - \frac{1}{2} + \frac{\frac{1}{4}}{2!} - \frac{\frac{1}{8}}{3!} + \frac{\frac{1}{16}}{4!} + \ldots \\
&= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} + \ldots \\
&\approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{48}{48} - \frac{24}{48} + \frac{6}{48} - \frac{1}{48} = \boxed{\frac{29}{48}} \leftarrow \text{estimate}
\end{aligned}
$$

Using ASET we can estimate the full sum using only the first four terms with error *at most* (the first neglected term in absolute value) $\frac{1}{384} < \frac{1}{100}$ as desired.

28. Estimate sin(1) with error less than $\frac{1}{1000}$. Justify. Hint: $7! = 5040$

Recall: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$\begin{aligned} \sin 1 &= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots \\ &= 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040} + \dots \\ &\approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{120}{120} - \frac{20}{120} + \frac{1}{120} = \boxed{\frac{101}{120}} \leftarrow \text{estimate} \end{aligned}$$

Using ASET we can estimate the full sum using only the first four terms with error *at most* (the first neglected term in absolute value) $\frac{1}{5040} < \frac{1}{1000}$ as desired.

29. Estimate $\arctan\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$. Justify.

Recall: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

$$\begin{aligned} \arctan\left(\frac{1}{2}\right) &= \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \dots \\ &= \frac{1}{2} - \frac{\frac{1}{8}}{3} + \frac{\frac{1}{32}}{5} - \dots \\ &= \frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \dots \\ &\approx \frac{1}{2} - \frac{1}{24} = \frac{12}{24} - \frac{1}{24} = \boxed{\frac{11}{24}} \leftarrow \text{estimate} \end{aligned}$$

Using ASET we can estimate the full sum using only the first four terms with error *at most* (the first neglected term in absolute value) $\frac{1}{160} < \frac{1}{100}$ as desired.