Math 121 Extra Partial Fraction Examples

We will learn to integrate some Rational Functions. Our main, first, examples will focus on decomposing *Proper* Rational Functions $R(x) = \dfrac{P(x)}{Q(x)}$ where Degree$P(x)$ <Degree$Q(x)$ using Partial Fractions algebra. We will mainly focus on two cases for Proper examples. First, when the denominator factors into two distinct linear pieces OR, second, when the denominator factors into one linear and one Quadratic Irreducible factor each.

Recall $ax^2 + bx + c$ is called Quadratic Irreducible if the Discriminant $b^2 - 4ac < 0$.

We will demonstrate some Optional Examples (for fun) at the end. Those will be more complicated problems that might require extra Complete the Square algebra on the Quadratic Irreducible factor. We will also demonstrate one optional example for an Improper Rational Function. Recall, that a Rational Function is called *Improper* if $\deg P(x) \geq degQ(x)$.

1. An example with two distinct Linear factors in the denominator

$$\begin{aligned}
\int \frac{8}{x^2 - 2x - 15}\, dx &= \int \frac{8}{(x-5)(x+3)}\, dx \\
&\overset{\text{PFD}}{=} \int \frac{1}{x-5} - \frac{1}{x+3}\, dx \\
&= \boxed{\ln|x-5| - \ln|x+3| + C}
\end{aligned}$$

Note that the integrand is a proper rational function because the degree of the numerator is strictly less than that of the denominator.

We use the following Partial Fractions Decomposition: PFD

$$\frac{1}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

Clearing the denominator yields:

$1 = A(x+3) + B(x-5)$
$1 = Ax + 3A + Bx - 5B$
$1 = (A+B)x + (3A - 5B)$
so that $\boxed{A + B = 0}$ and $\boxed{3A - 5B = 1}$

Solve for $\boxed{A = 1}$ and $\boxed{B = -1}$

2. Another example with two distinct Linear factors in the denominator

$$\int \frac{16-x}{x^2-5x-14}\,dx = \int \frac{16-x}{(x-7)(x+2)}\,dx$$

$$\overset{\text{PFD}}{=} \int \frac{1}{x-7} - \frac{2}{x+2}\,dx$$

$$= \boxed{\ln|x-7| - 2\ln|x+2| + C}$$

We use the following Partial Fractions Decomposition: PFD

$$\frac{16-x}{(x-7)(x+2)} = \frac{A}{x-7} + \frac{B}{x+2}$$

Clearing the denominator yields:

$16 - x = A(x+2) + B(x-7)$
$-x + 16 = Ax + 2A + Bx - 7B$
$-x + 16 = (A+B)x + (2A - 7B)$
so that $\boxed{A + B = -1}$ and $\boxed{2A - 7B = 16}$

Solve for $\boxed{A = 1}$ and $\boxed{B = -2}$

3. Another example with two distinct Linear factors in the denominator

$$\int \frac{2x-23}{x^2-3x-4}\,dx = \int \frac{2x-23}{(x+1)(x-4)}\,dx$$

$$\overset{\text{PFD}}{=} \int \frac{5}{x+1} - \frac{3}{x-4}\,dx$$

$$= \boxed{5\ln|x+1| - 3\ln|x-4| + C}$$

We use the following Partial Fractions Decomposition: PFD

$$\frac{2x-23}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

Clearing the denominator yields:

$2x - 23 = A(x-4) + B(x+1)$
$2x - 23 = Ax - 4A + Bx + B$
$2x - 23 = (A+B)x + (-4A + B)$
so that $\boxed{A + B = 2}$ and $\boxed{-4A + B = -23}$

Solve for $\boxed{A = 5}$ and $\boxed{B = -3}$

4. An example where the denominator factors into one Linear and one Quadratic Irreducible piece.

Compute $\displaystyle\int \frac{1}{(x-2)(x^2+1)}\,dx$

Note that the integrand is also a proper rational function because the degree of the numberator is strictly less than that of the denominator.

We use the following Partial Fractions decomposition:

$$\frac{1}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$$

Clearing the denominator yields:

$1 = A(x^2+1) + (Bx+C)(x-2)$
$1 = Ax^2 + A + Bx^2 + Cx - 2Bx - 2C$
$1 = (A+B)x^2 + (C-2B)x + A - 2C$
so that $\boxed{A+B=0}$ and $\boxed{C-2B=0}$ and $\boxed{A-2C=1}$

Solve for $\boxed{A=\frac{1}{5}}$ and $\boxed{B=-\frac{1}{5}}$ and $\boxed{C=-\frac{2}{5}}$

Finally, we have decomposed the original integrand. Hopefully this new, but equal, decomposition is *easier* to integrate.

$$\int \frac{1}{(x-2)(x^2+1)}\,dx = \int \frac{\frac{1}{5}}{x-2} + \frac{-\frac{1}{5}x-\frac{2}{5}}{x^2+1}\,dx \quad \bullet \text{ how to handle?}$$

$$= \frac{1}{5}\int \frac{1}{x-2}\,dx - \frac{1}{5}\int \frac{x}{x^2+1}\,dx - \frac{2}{5}\int \frac{1}{x^2+1}\,dx$$

- after split of integrals
- here we have some natural log terms (how?), and an "arctan"-ish term
- make sure that you understand how to compute these integrals quickly

$$= \frac{1}{5}\ln|x-2| - \frac{1}{5}\frac{\ln|x^2+1|}{2} - \frac{2}{5}\arctan x + C$$

$$= \boxed{\frac{1}{5}\ln|x-2| - \frac{1}{10}\ln|x^2+1| - \frac{2}{5}\arctan x + C}$$

5. Another example where the denominator factors into one Linear and one Quadratic Irreducible piece.

Compute $\displaystyle\int \frac{5x^2 - 10x + 11}{(x-3)(x^2+4)}\, dx$

Note that the integrand is also a proper rational function because the degree of the numberator is strictly less than that of the denominator.

We use the following Partial Fractions decomposition:

$$\frac{5x^2 - 10x + 11}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx + C}{x^2+4}$$

Clearing the denominator yields:

$5x^2 - 10x + 11 = A(x^2 + 4) + (Bx + C)(x - 3)$
$5x^2 - 10x + 11 = Ax^2 + 4A + Bx^2 + Cx - 3Bx - 3C$
$5x^2 - 10x + 11 = (A + B)x^2 + (C - 3B)x + 4A - 3C$
so that $\boxed{A + B = 5}$ and $\boxed{C - 3B = -10}$ and $\boxed{4A - 3C = 11}$

Solve for $\boxed{A = 2}$ and $\boxed{B = 3}$ and $\boxed{C = -1}$

Finally, we have decomposed the original integrand.

$$\int \frac{5x^2 - 10x + 11}{(x-3)(x^2+4)}\, dx = \int \frac{2}{x-3} + \frac{3x-1}{x^2+4}\, dx \quad \bullet \text{ how to handle?}$$

$$= 2\int \frac{1}{x-3}\, dx - 3\int \frac{x}{x^2+4}\, dx - \int \frac{1}{x^2+4}\, dx$$

- after split of integrals
- here we have some natural log terms (how?), and an "arctan"-ish term
- make sure that you understand how to compute these integrals quickly

$$= \boxed{2\ln|x-3| + \frac{3\ln|x^2+1|}{2} - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

For Fun! Here the Quadratic Irreducible piece requires *Complete the Square* algebra.

6. Compute $\displaystyle\int \frac{1}{(x-1)(x^2+x+1)}\, dx$

Note that the integrand is a proper rational function because the degree of the numberator is strictly less than that of the denominator. We will *Complete the square* on the irreducible factor in order to convert it into an "almost" arctan integral.

For now, we use the following Partial Fractions decomposition:

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

Clearing the denominator yields:

$1 = A(x^2+x+1) + (Bx+C)(x-1)$
$1 = Ax^2 + Ax + A + Bx^2 + Cx - Bx - C$
$1 = (A+B)x^2 + (A-B+C)x + A - C$
so that $\boxed{A+B=0}$ and $\boxed{A-B+C=0}$ and $\boxed{A-C=1}$

Solve for $\boxed{A=\frac{1}{3}}$ and $\boxed{B=-\frac{1}{3}}$ and $\boxed{C=-\frac{2}{3}}$

Finally, we have decomposed the original integrand. Hopefully this new, but equal, decomposition is *easier* to integrate.

$$\int \frac{1}{(x-1)(x^2+x+1)}\,dx = \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx$$

$$= \int \frac{\frac{1}{3}}{x-1}\,dx - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

• complete the square on the quadratic irreducible second term

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{\left(u-\frac{1}{2}\right)+2}{u^2+\frac{3}{4}}\,du$$ • see subst. below

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{u+\frac{3}{2}}{u^2+\frac{3}{4}}\,du$$ • simplify

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du - \frac{1}{3}\int \frac{\frac{3}{2}}{u^2+\frac{3}{4}}\,du$$

• after split of integrals
• here we have some natural log terms (how?), and an "arctan"-ish term

$$= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{u}{u^2+\frac{3}{4}}\,du - \frac{1}{3}\int \frac{\frac{3}{2}}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,du$$

• make sure that you understand how to compute these integrals quickly

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\frac{\ln|u^2+\frac{3}{4}|}{2} - \frac{1}{3}\left(\frac{3}{2}\right)\left(\frac{2}{\sqrt{3}}\right)\arctan\left(\frac{u}{\left(\frac{\sqrt{3}}{2}\right)}\right) + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|u^2+\frac{3}{4}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{3}}\right) + C$$

$$= \boxed{\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left|x^2+x+1\right| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

We ("invertedly") substituted above $\boxed{u = x+\frac{1}{2} \Rightarrow x = u-\frac{1}{2} \qquad du = dx}$

For Fun again! Here the Quadratic Irreducible piece requires *Complete the Square* algebra and a "reverse" u-sub.

7. Compute $\displaystyle\int \frac{x+13}{x(x^2+4x+13)}\,dx$

Note that the integrand is already a proper rational function. The denominator is already factored into a linear factor and a quadratic irreducible factor. (why is that irreducible?) We use the following Partial Fractions decomposition:

$$\frac{x+13}{x(x^2+4x+13)} = \frac{A}{x} + \frac{Bx+C}{x^2+4x+13}$$

Clearing the denominator yields:

$x+13 = A(x^2+4x+13) + (Bx+C)x$
$x+13 = Ax^2+4Ax+13A+Bx^2+Cx$
$x+13 = (A+B)x^2+(4A+C)x+13A$
so that $\boxed{A+B=0}$ and $\boxed{4A+C=1}$ and $\boxed{13A=13}$

Solve for $\boxed{A=1}$ and $\boxed{B=-1}$ and $\boxed{C=-3}$

Finally, we have decomposed the original integrand. Hopefully this new, but equal, decomposition is *easier* to integrate.

$$\frac{x+13}{x(x^2+4x+13)} = \frac{1}{x} + \frac{-x-3}{x^2+4x+13}$$

Now,

$$= \int \frac{x+13}{x(x^2+4x+13)}\,dx = \int \frac{1}{x} + \frac{-x-3}{x^2+4x+13}\,dx = \int \frac{1}{x} - \frac{x+3}{x^2+4x+13}\,dx$$

- complete the square on the quadratic irreducible second term

$$= \int \frac{1}{x} - \frac{x+3}{(x+2)^2+9}\,dx = \int \frac{1}{x}\,dx - \int \frac{(u-2)+3}{u^2+9}\,du \quad \bullet \text{ see subst. below}$$

$$= \int \frac{1}{x}\,dx - \int \frac{u+1}{u^2+9}\,du = \int \frac{1}{x}\,dx - \int \frac{u}{u^2+9}\,du - \int \frac{1}{u^2+9}\,du \quad \bullet \text{ after split of integrals}$$

- here we have some natural log terms, and an "arctan"-ish term

- make sure that you understand how to compute the last two integrals quickly

$$= \ln|x| - \frac{1}{2}\ln|u^2+9| - \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \boxed{\ln|x| - \frac{1}{2}\ln|(x+2)^2+9| - \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C}$$

We ("invertedly") substituted above

$$\boxed{\begin{array}{l} u = x + 2 \Rightarrow x = u - 2 \\ \\ du = dx \end{array}}$$

Just for FUN! Improper Rational Function case...OPTIONAL.

8. Compute $\displaystyle\int \frac{x^2+7}{x^2+2x+10}\,dx$

Note that the integrand is an improper rational function because the degree of the numberator is *not* strictly less than that of the denominator. We apply long division of polynomials to write the integrand as a (simple) polynomial and a proper rational piece. That is, the degree of the numerator is strictly less than that of the denominator.

Long division yields:

$$\begin{array}{r} 1 \\ x^2+2x+10\overline{)x^2+7} \\ -(x^2+2x+10) \\ \hline -2x-3 \end{array}$$

That means $x^2+7 = (x^2+2x+10)(1) + (-2x-3)$. Here the remainder term is $-2x-3$.
Dividing both sides by $x^2+2x+10$ yields $\displaystyle\frac{x^2+7}{x^2+2x+10} = 1 + (-2x-3)$

Now $\displaystyle\int \frac{x^2+7}{x^2+2x+10}\,dx = \int 1 + \frac{-2x-3}{x^2+2x+10}\,dx = \int 1 - \frac{2x+3}{x^2+2x+10}\,dx$

The second piece contains a quadratic irreducible, so partial fractions will not be helpful here. *Complete the square* on that irreducible piece to convert it into an "almost" arctan integral.

$$= \int 1 - \frac{2x+3}{(x+1)^2+9}\,dx = \int 1\,dx - \int \frac{2(u-1)+3}{u^2+9}\,du \quad \bullet \text{ see subst. below}$$

$$= \int 1\,dx - \int \frac{2u-2+3}{u^2+9}\,du = \int 1\,dx - \int \frac{2u+1}{u^2+9}\,du \quad \bullet \text{ simplify}$$

$$= \int 1\,dx - \int \frac{2u}{u^2+9}\,du - \int \frac{1}{u^2+9}\,du \quad \bullet \text{ after split of integrals}$$

- here we have a simple term, a natural log term, and an "arctan"-ish term
- make sure that you understand how to compute the last two integrals quickly

$$= x - \ln|u^2+9| - \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C = \boxed{x - \ln|(x+1)^2+9| - \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C}$$

We ("invertedly") substituted above

$$\boxed{\begin{array}{l} u = x + 1 \Rightarrow x = u - 1 \\ \\ du = dx \end{array}}$$