Natural Logarithm Function $y = \ln x$ Review

In this handout, we will review the Natural Logarithm Function $y = \ln x$ by studying its related

1. Function properties
2. Limits
3. Derivatives
4. Integrals

Start by reviewing the Natural Exponential Function $y = e^x$. It is indeed a one-to-one function, since it is a strictly increasing function (why?).

Recall: The exponential function
- **NEVER** yield the output value $y = 0$
- **NEVER** yield a negative output value

Graph:

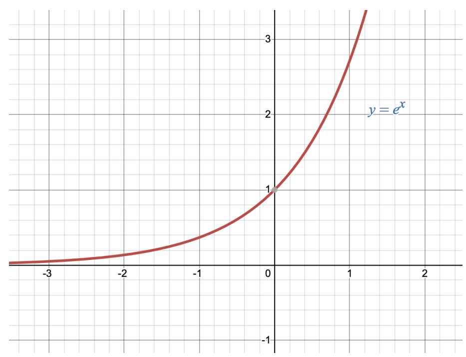

Figure 1: Natural Exponential Function $y = e^x$

Goal: To find an Inverse function for the Natural Exponential function $y = e^x$, that is, find a **reversing** function to solve an equation like $e^{1-3x} = 9$

Note: There are different approaches to exploring the Logarithm, but we will focus on the Inverse of $y = e^x$.

Definition: The Natural Logarithm Function $y = \log_e x$ is defined as the unique y-value such that $e^y = x$.

Think: the logarithm base e is "the exponent y for which e is raised to in order to get back the original input x". We can refer to this as a ① − ② − ③ Memory-Aid Rule. That is, ① raised to the ② equals ③

$$\log_{e_{①}} x_{③} \hookrightarrow = y_{②} \iff e_{①}^{y_{②}} = x_{③}$$

Shorthand: Set $\ln x = \log_e x$

$$\ln x = y \iff e^y = x$$

Recall that the graph of Inverse Functions are mirror symmetrical flips across the line $y = x$.
Graph:

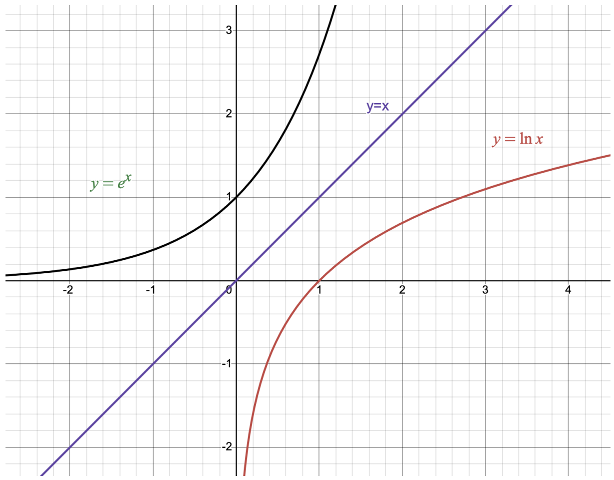

Figure 2: Natural Logarithm Function $y = \ln x$

Let's study some properties of this Natural Logarithm Function $y = \ln x$.

Domain$= (0, \infty) \overset{\text{or}}{=} \{x : x > 0\}$

Recall, the *Domain* of a function is the collection of all possible input values which yield a finite output or the values for which the function is defined. Here, $\ln x$ is only defined for strictly positive input values $x > 0$

Recall: The logarithm function
- **NEVER** have an input value $y = 0$
- **NEVER** have a negative input value

Range$= (-\infty, \infty) = \mathbb{R}$

Recall, the *Range* of a function is the collection of all possible output values for a given function. Here $\ln x$ yields all possible output values.

Note: As Inverse Functions it makes sense, from their graphs/definitions, that

Domain e^x	$=$	Range $\ln x$
Domain $\ln x$	$=$	Range e^x

Value(s): $\ln 1 = 0$

We can read this off the graph, but we can also think about the defined value.
$\ln 1 = \log_e(1) = ?$ which means in reverse that $e^? = 1$ so $? = 0$. Think: ①−②−③ Memory-Aid.

WARNING: $\ln 0$ is undefined, that is, there is no value y such that $e^y = 0$ since the exponential function is never output 0.

Inverses:

$$\begin{array}{|ll|}\hline \ln e^x = x & \text{for all x} \\ e^{\ln x} = x & \text{for } x > 0 \\ \hline\end{array}$$

Note: these inverse properties state that the log and exponential invert each other, they literally undo or unwind the other function value. They don't just "cancel".

Example: Simplify $\ln\left(e^7\right) = \cancel{\ln}\left(\cancel{e}^7\right) = 7$

Example: Simplify $e^{\ln 3} = \cancel{e}^{\cancel{\ln}3} = 3$

Example: Solve $e^{1-3x} = 9$ by taking logs of both sides, $\ln\left(e^{1-3x}\right) = \ln 9$ with $1 - 3x = \ln 9$ solved to $x = \dfrac{(\ln 9) - 1}{-3}$

Example: Solve $\ln(7x + 4) = 8$ by applying exponentials to both sides, $e^{\ln(7x+4)} = e^8$ with $7x + 4 = e^8$ solved to $x = \dfrac{e^8 - 4}{7}$

Algebra:

Rule	Tip
$\ln a + \ln b = \ln(a \cdot b)$	sum of the logs equals the log of the product warning: **NOT** $\ln a \cdot \ln b$
$\ln a - \ln b = \ln\left(\dfrac{a}{b}\right)$	difference of the logs equals the log of the quotient warning: **NOT** $\dfrac{\ln a}{\ln b}$
$\ln\left(a^b\right) = b \cdot \ln a$	power rule, constant can move down (from) or up (to) the exponent watch the parentheses here

Warning: $\ln(a \pm b)$ and $\dfrac{\ln a}{\ln b}$ do **not** simplify

Example: Simplify $\ln 2 + \ln 2 + \ln 2 = 3 \overset{\curvearrowright}{\ln}\, 2 = \ln\left(2^3\right) = \ln 8$

Example: Simplify $\ln 8 - \ln 2 = \ln\left(\dfrac{8}{2}\right) = \ln 4$

Example: Simplify $e^{-3\ln 2} = e^{\ln\left(2^{-3}\right)} = \cancel{e}^{\cancel{\ln}\left(2^{-3}\right)} = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

Limits: Study Graph

Limit	Tip
$\lim\limits_{x\to\infty} \ln x = \infty$	as input values grow uncontrollably large, the $\ln x$ output grows to ∞
$\lim\limits_{x\to 0^+} \ln x = -\infty$	as input shrinks towards 0 from the right, the $\ln x$ shoots to $-\infty$

Example: $\lim\limits_{x\to5^+} \ln(x-5) = \lim\limits_{x\to5^+} \ln\left(\cancelto{0^+}{\cancelto{5^+}{x}-5}\right) = -\infty$ The arrows help justify the size argument(s).

Ex: $\lim\limits_{x\to8^-} \ln|x-8| = \lim\limits_{x\to8^-} \ln\left|\cancelto{0^+}{\cancelto{0^-}{\cancelto{8^-}{x}-8}}\right| = -\infty$

Derivatives:

Derivative	Tip
$\dfrac{d}{dx}\ln x = \dfrac{1}{x}$	derivative of log flips the input variable x
$\dfrac{d}{dx}\ln\left(u(x)\right) = \dfrac{1}{u(x)} \cdot u'(x)$	**CHAIN RULE** flips the original input chunk... times the derivative of the *inside nested* function

Recall: the Derivative Chain Rule can be written as

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) = \underbrace{f'(g(x))}_{\substack{\text{deriv of outside} \\ \text{leave inside}}} \cdot \underbrace{g'(x)}_{\text{deriv of inside}}}$$

Let us justify why the derivative of $\ln x$ equals $\dfrac{1}{x}$. That is, Prove: $\boxed{\dfrac{d}{dx}\ln x = \dfrac{1}{x}}$

Let $y = \ln x$

Invert $e^y = x$

Differentiate $\dfrac{d}{dx}\left(e^y\right) = \dfrac{d}{dx}(x)$ yielding $e^y \dfrac{dy}{dx} = 1$

Finally, Solve $\dfrac{dy}{dx} = \dfrac{1}{e^y} = \dfrac{1}{x}$

Example: $\dfrac{d}{dx}\ln(5+8x) = \dfrac{1}{5+8x}\cdot 8 = \dfrac{8}{1+5x}$

Example: $\dfrac{d}{dx}\sqrt{3+\ln x} = \dfrac{1}{2\sqrt{3+\ln x}}\cdot\left(\dfrac{1}{x}\right) = \dfrac{1}{2x\sqrt{3+\ln x}}$

Example: $\dfrac{d}{dx}\ln\left(7+e^{6x}\right) = \dfrac{1}{7+e^{6x}}\cdot\left(6e^{6x}\right) = \dfrac{6e^{6x}}{7+e^{6x}}$

Note that the order of composed function matters. The following are not the same.

Example: $\dfrac{d}{dx}\ln\left(\sin x\right) = \dfrac{1}{\sin x}\cdot\cos x$

Example: $\dfrac{d}{dx}\sin(\ln x) = \cos(\ln x)\cdot\dfrac{1}{x}$

Here again, the following are not the same.

Example: $\dfrac{d}{dx}\left(\ln x\right)^3 = 3\left(\ln x\right)^2\dfrac{1}{x} = \dfrac{3\left(\ln x\right)^2}{x}$

Example: $\dfrac{d}{dx}\ln\left(x^3\right) = \dfrac{1}{x^3}\cdot\left(3x^2\right) = \dfrac{3}{x}$

Note: there is another way to compute this by simplifying first, then differentiating

Again: $\dfrac{d}{dx}\ln\left(x^3\right) = \dfrac{d}{dx}3\ln\left(x\right) = 3\left(\dfrac{1}{x}\right) = \dfrac{3}{x}$

Integrals:

Integral	Tip
$\displaystyle\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	antiderivative of $\dfrac{1}{x}$ is the natural log

WARNING: Be careful to use the Absolute Value, to make sure the logarithm does not get handed negative values

IMPORTANT: We now have the function which is the Antiderivative of $\dfrac{1}{x}$

Recall: the classic Power Rule $\boxed{\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1}$

If we try to apply this Power Rule to $\dfrac{1}{x}$ then we would get division by 0, undefined.

Helpful **WARNING/INCORRECT**: $\int \frac{1}{x}\,dx \neq \frac{x^0}{0} + C$

Note: be careful not to punch every integral with a denominator with the logarithm. That is, be careful to **only** use the log antiderivative rule for 1 over x **to the exact power 1**. If there is any other power of x, then we use the Power Rule above.

Helpful **WARNING/INCORRECT**: $\int \frac{1}{\sqrt{x}}\,dx \neq \ln\sqrt{x} + C$

Instead, using the correct power rule

$$\int \frac{1}{\sqrt{x}}\,dx \overset{\text{prep}}{=} \int x^{-\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{x} + C$$

We might also be tempted to use the logarithm on the following

Helpful **WARNING/INCORRECT**: $\int \frac{1}{e^{6x}}\,dx \neq \ln|e^{6x}| + C$

Instead, using the correct k-Rule

$$\int \frac{1}{e^{6x}}\,dx \overset{\text{prep}}{=} \int e^{-6x}\,dx = -\frac{1}{6}e^{-6x} + C = -\frac{1}{6e^{6x}} + C$$

INDEFINITE Integrals with u-substitution: Always remember to add +C right away, as soon as you compute the Most General Antiderivative. The original variable always reappears when we re-substitute back for u.

Recall: u-substitution is a temporary convenience that hides a nested, meaty chunk of your integrand to first simplify the integral, and second, to match the derivative chunk, all with the overall goal of reversing the Chain Rule.

Example: $\int \frac{e^{3x}}{5+e^{3x}} = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \boxed{\frac{1}{3}\ln|5+e^{3x}| + C}$

$$\begin{aligned} u &= 5 + e^{3x} \\ du &= 3e^{3x}\,dx \\ \tfrac{1}{3}du &= e^{3x}dx \end{aligned}$$

Example: $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = \boxed{-\ln|\cos x| + C}$

$$\begin{aligned} u &= \cos x \\ du &= -\sin x dx \\ -du &= \sin x dx \end{aligned}$$

Note: Many different integrals lead to the same u-sub integral leading to Log.

Example: $\displaystyle\int \frac{1}{\sqrt{x}\left(8+\sqrt{x}\right)}\,dx = 2\int \frac{1}{u}\,du = 2\ln|u| + C = 2\ln|8+\sqrt{x}| + C$

$$\boxed{\begin{aligned} u &= 8+\sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \\ 2du &= \frac{1}{\sqrt{x}}\,dx \end{aligned}}$$

Example:

$$\int \frac{1}{x\left(9+\ln x\right)^3}\,dx = \int \frac{1}{u^3}\,du \overset{\text{prep}}{=} \int u^{-3}\,du = \frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{2u^2} + C = \boxed{-\frac{1}{2\left(9+\ln x\right)^2} + C}$$

$$\boxed{\begin{aligned} u &= 9+\ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$$

IMPORTANT: Notice for u-sub with logarithms, sometimes the log *acts as* the u and sometimes the denominator *holds* the u with its power exactly 1, antidifferentiating to log.

QUESTION: How do we know which method of integration to use, u-substitution or Algebra or both? Here are two examples that model the different options. Generally, try u-sub and if the derivative du does not match, then you can try Algebra as a (good) back-up plan. Let us study two similar examples with different integration methods ...

Example: u-substitution works

$$\int \frac{x^6}{2-x^7}\,dx = -\frac{1}{7}\int \frac{1}{u}\,du = -\frac{1}{7}\ln|u| + C = \boxed{-\frac{1}{7}\ln|2-x^7| + C}$$

$$\boxed{\begin{aligned} u &= 2-x^7 \\ du &= -7x^6\,dx \\ -\tfrac{1}{7}du & \quad x^6 dx \end{aligned}}$$

Example: Algebra plus log rule works

$$\int \frac{2-x^6}{x^7}\,dx \overset{\text{split}}{=} \int \frac{2}{x^7} - \frac{x^6}{x^7}\,dx \overset{\text{prep}}{=} \int 2x^{-7} - \frac{1}{x}\,dx$$

$$= 2\left(\frac{x^{-6}}{-6}\right) - \ln|x| + C = \boxed{-\frac{1}{3x^6} - \ln|x| + C}$$

DEFINITE Integrals: Recall, you must change (or temporarily mark) your Limits of integration. The variables and Limits of Integration change *simultaneously*. Once you *switch* your Limits of Integration to u-values, then the original variable never reappears.

Example:

$$\int_2^3 \frac{1}{5-4x}\,dx \;= -\frac{1}{4}\int_{-3}^{-7}\frac{1}{u}\,du = -\frac{1}{4}\ln|u|\Big|_{-3}^{-7}$$

$$= -\frac{1}{4}\left(\ln|-7| - \ln|-3|\right) = \boxed{-\frac{1}{4}\ln\left(\frac{7}{3}\right)}$$

$$\boxed{\begin{aligned} u &= 5-4x \\ du &= -4\,dx \\ -\tfrac{1}{4}du &= dx \end{aligned}} \text{ and } \boxed{\begin{aligned} &x = 2 \Rightarrow u = 5-8 = -3 \\ &x = 3 \Rightarrow u = 5-12 = -7 \end{aligned}}$$

Note: Without the Absolute Value bars, we would have undefined log values of negative numbers

Example:

$$\int_0^{\ln 3} \frac{e^{2x}}{1+e^{2x}}\,dx \;= \frac{1}{2}\int_2^{10}\frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_2^{10} = \frac{1}{2}\left(\ln|10| - \ln|2|\right)$$

$$= \frac{1}{2}\ln\left(\frac{10}{2}\right) = \boxed{\frac{1}{2}\ln 5 + C}$$

$$\boxed{\begin{aligned} u &= 1+e^{2x} \\ du &= 2e^{2x}dx \\ \tfrac{1}{2}du &= e^{2x}dx \end{aligned}} \text{ and } \boxed{\begin{aligned} &x = 0 \Rightarrow u = 1+e^0 = 1+1 = 2 \\ &x = \ln 3 \Rightarrow u = 1+e^{2\ln 3} = 1+e^{\ln(3^2)} = 10 \end{aligned}}$$

Example:

$$\int_{\ln 3}^{\ln 8} \frac{e^x}{\sqrt{1+e^x}}\,dx \;= \int_4^9 \frac{1}{\sqrt{u}}\,du = \int_4^9 u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_4^9 = 2\sqrt{u}\Big|_4^9$$

$$= 2\left(\sqrt{9}-\sqrt{4}\right) = 2(3-2) = \boxed{2}$$

$$\boxed{\begin{aligned} u &= 1+e^x \\ du &= e^x dx \end{aligned}} \text{ and } \boxed{\begin{aligned} &x = \ln 3 \Rightarrow u = 1+e^{\ln 3} = 1+3 = 4 \\ &x = \ln 8 \Rightarrow u = 1+e^{\ln 8} = 1+8 = 9 \end{aligned}}$$