EXAMPLE 5 Find $\dfrac{d}{dx}[\tanh^{-1}(\sin x)]$.

SOLUTION Using Table 6 and the Chain Rule, we have

$$\frac{d}{dx}[\tanh^{-1}(\sin x)] = \frac{1}{1-(\sin x)^2}\frac{d}{dx}(\sin x)$$

$$= \frac{1}{1-\sin^2 x}\cos x = \frac{\cos x}{\cos^2 x} = \sec x$$

EXAMPLE 6 Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x^2}}$.

SOLUTION Using Table 6 (or Example 4) we know that an antiderivative of $1/\sqrt{1+x^2}$ is $\sinh^{-1}x$. Therefore

$$\int_0^1 \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}x\Big]_0^1$$

$$= \sinh^{-1} 1$$

$$= \ln\left(1+\sqrt{2}\right) \qquad \text{(from Equation 3)}$$

6.7 EXERCISES

1–6 Find the numerical value of each expression.

1. (a) $\sinh 0$ (b) $\cosh 0$
2. (a) $\tanh 0$ (b) $\tanh 1$
3. (a) $\cosh(\ln 5)$ (b) $\cosh 5$
4. (a) $\sinh 4$ (b) $\sinh(\ln 4)$
5. (a) $\operatorname{sech} 0$ (b) $\cosh^{-1} 1$
6. (a) $\sinh 1$ (b) $\sinh^{-1} 1$

7–19 Prove the identity.

7. $\sinh(-x) = -\sinh x$
(This shows that sinh is an odd function.)
8. $\cosh(-x) = \cosh x$
(This shows that cosh is an even function.)
9. $\cosh x + \sinh x = e^x$
10. $\cosh x - \sinh x = e^{-x}$
11. $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
12. $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
13. $\coth^2 x - 1 = \operatorname{csch}^2 x$
14. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$
15. $\sinh 2x = 2 \sinh x \cosh x$
16. $\cosh 2x = \cosh^2 x + \sinh^2 x$
17. $\tanh(\ln x) = \dfrac{x^2-1}{x^2+1}$
18. $\dfrac{1+\tanh x}{1-\tanh x} = e^{2x}$
19. $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$
(n any real number)

20. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

21. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x.

22. (a) Use the graphs of sinh, cosh, and tanh in Figures 1–3 to draw the graphs of csch, sech, and coth.
(b) Check the graphs that you sketched in part (a) by using a graphing device to produce them.

23. Use the definitions of the hyperbolic functions to find each of the following limits.
(a) $\lim_{x\to\infty} \tanh x$ (b) $\lim_{x\to-\infty} \tanh x$
(c) $\lim_{x\to\infty} \sinh x$ (d) $\lim_{x\to-\infty} \sinh x$
(e) $\lim_{x\to\infty} \operatorname{sech} x$ (f) $\lim_{x\to\infty} \coth x$
(g) $\lim_{x\to 0^+} \coth x$ (h) $\lim_{x\to 0^-} \coth x$
(i) $\lim_{x\to-\infty} \operatorname{csch} x$ (j) $\lim_{x\to\infty} \dfrac{\sinh x}{e^x}$

24. Prove the formulas given in Table 1 for the derivatives of the functions (a) cosh, (b) tanh, (c) csch, (d) sech, and (e) coth.

25. Give an alternative solution to Example 3 by letting $y = \sinh^{-1}x$ and then using Exercise 9 and Example 1(a) with x replaced by y.

26. Prove Equation 4.

27. Prove Equation 5 using (a) the method of Example 3 and (b) Exercise 18 with x replaced by y.

28. For each of the following functions (i) give a definition like those in (2), (ii) sketch the graph, and (iii) find a formula similar to Equation 3.
(a) csch^{-1} (b) sech^{-1} (c) $\coth^{-1}$

29. Prove the formulas given in Table 6 for the derivatives of the following functions.
(a) $\cosh^{-1}$ (b) $\tanh^{-1}$ (c) csch^{-1}
(d) sech^{-1} (e) $\coth^{-1}$

30–45 Find the derivative. Simplify where possible.

30. $f(x) = e^x \cosh x$

31. $f(x) = \tanh \sqrt{x}$

32. $g(x) = \sinh^2 x$

33. $h(x) = \sinh(x^2)$

34. $F(t) = \ln(\sinh t)$

35. $G(t) = \sinh(\ln t)$

36. $y = \operatorname{sech} x\,(1 + \ln \operatorname{sech} x)$

37. $y = e^{\cosh 3x}$

38. $f(t) = \dfrac{1 + \sinh t}{1 - \sinh t}$

39. $g(t) = t \coth \sqrt{t^2 + 1}$

40. $y = \sinh^{-1}(\tan x)$

41. $y = \cosh^{-1}\sqrt{x}$

42. $y = x \tanh^{-1} x + \ln \sqrt{1 - x^2}$

43. $y = x \sinh^{-1}(x/3) - \sqrt{9 + x^2}$

44. $y = \operatorname{sech}^{-1}(e^{-x})$

45. $y = \coth^{-1}(\sec x)$

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46. Show that $\dfrac{d}{dx} \sqrt[4]{\dfrac{1 + \tanh x}{1 - \tanh x}} = \frac{1}{2} e^{x/2}$.

47. Show that $\dfrac{d}{dx} \arctan(\tanh x) = \operatorname{sech} 2x$.

48. The Gateway Arch in St. Louis was designed by Eero Saarinen and was constructed using the equation

$$y = 211.49 - 20.96 \cosh 0.03291765x$$

for the central curve of the arch, where x and y are measured in meters and $|x| \leq 91.20$.
(a) Graph the central curve.
(b) What is the height of the arch at its center?
(c) At what points is the height 100 m?
(d) What is the slope of the arch at the points in part (c)?

49. If a water wave with length L moves with velocity v in a body of water with depth d, then

$$v = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi d}{L}\right)}$$

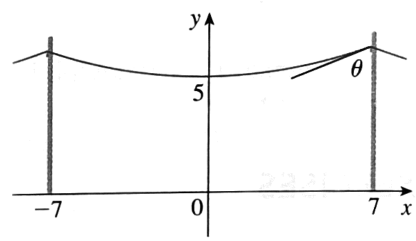
where g is the acceleration due to gravity. (See Figure 5.) Explain why the approximation

$$v \approx \sqrt{\frac{gL}{2\pi}}$$

is appropriate in deep water.

50. A flexible cable always hangs in the shape of a catenary $y = c + a \cosh(x/a)$, where c and a are constants and $a > 0$ (see Figure 4 and Exercise 52). Graph several members of the family of functions $y = a \cosh(x/a)$. How does the graph change as a varies?

51. A telephone line hangs between two poles 14 m apart in the shape of the catenary $y = 20 \cosh(x/20) - 15$, where x and y are measured in meters.
(a) Find the slope of this curve where it meets the right pole.
(b) Find the angle θ between the line and the pole.



52. Using principles from physics it can be shown that when a cable is hung between two poles, it takes the shape of a curve $y = f(x)$ that satisfies the differential equation

$$\frac{d^2y}{dx^2} = \frac{\rho g}{T} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where ρ is the linear density of the cable, g is the acceleration due to gravity, T is the tension in the cable at its lowest point, and the coordinate system is chosen appropriately. Verify that the function

$$y = f(x) = \frac{T}{\rho g} \cosh\left(\frac{\rho g x}{T}\right)$$

is a solution of this differential equation.

53. A cable with linear density $\rho = 2$ kg/m is strung from the tops of two poles that are 200 m apart.
(a) Use Exercise 52 to find the tension T so that the cable is 60 m above the ground at its lowest point. How tall are the poles?
(b) If the tension is doubled, what is the new low point of the cable? How tall are the poles now?

54. A model for the velocity of a falling object after time t is

$$v(t) = \sqrt{\frac{mg}{k}} \tanh\left(t \sqrt{\frac{gk}{m}}\right)$$

where m is the mass of the object, $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity, k is a constant, t is measured in seconds, and v in m/s.

(a) Calculate the terminal velocity of the object, that is, $\lim_{t\to\infty} v(t)$.

(b) If a person falls from a building, the value of the constant k depends on his or her position. For a "belly-to-earth" position, $k = 0.515$ kg/s, but for a "feet-first" position, $k = 0.067$ kg/s. If a 60-kg person falls in belly-to-earth position, what is the terminal velocity? What about feet-first?

Source: L. Long et al., "How Terminal Is Terminal Velocity?" *American Mathematical Monthly* 113 (2006): 752–55.

55. (a) Show that any function of the form

$$y = A \sinh mx + B \cosh mx$$

satisfies the differential equation $y'' = m^2 y$.

(b) Find $y = y(x)$ such that $y'' = 9y$, $y(0) = -4$, and $y'(0) = 6$.

56. If $x = \ln(\sec\theta + \tan\theta)$, show that $\sec\theta = \cosh x$.

57. At what point of the curve $y = \cosh x$ does the tangent have slope 1?

58. Investigate the family of functions

$$f_n(x) = \tanh(n \sin x)$$

where n is a positive integer. Describe what happens to the graph of f_n when n becomes large.

59–67 Evaluate the integral.

59. $\int \sinh x \cosh^2 x\, dx$

60. $\int \sinh(1 + 4x)\, dx$

61. $\int \dfrac{\sinh\sqrt{x}}{\sqrt{x}}\, dx$

62. $\int \tanh x\, dx$

63. $\int \dfrac{\cosh x}{\cosh^2 x - 1}\, dx$

64. $\int \dfrac{\operatorname{sech}^2 x}{2 + \tanh x}\, dx$

65. $\int_4^6 \dfrac{1}{\sqrt{t^2 - 9}}\, dt$

66. $\int_0^1 \dfrac{1}{\sqrt{16t^2 + 1}}\, dt$

67. $\int \dfrac{e^x}{1 - e^{2x}}\, dx$

68. Estimate the value of the number c such that the area under the curve $y = \sinh cx$ between $x = 0$ and $x = 1$ is equal to 1.

69. (a) Use Newton's method or a graphing device to find approximate solutions of the equation $\cosh 2x = 1 + \sinh x$.

(b) Estimate the area of the region bounded by the curves $y = \cosh 2x$ and $y = 1 + \sinh x$.

70. Show that the area of the shaded hyperbolic sector in Figure 7 is $A(t) = \frac{1}{2}t$. [*Hint:* First show that

$$A(t) = \tfrac{1}{2}\sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\, dx$$

and then verify that $A'(t) = \frac{1}{2}$.]

71. Show that if $a \neq 0$ and $b \neq 0$, then there exist numbers α and β such that $ae^x + be^{-x}$ equals either $\alpha \sinh(x + \beta)$ or $\alpha \cosh(x + \beta)$. In other words, almost every function of the form $f(x) = ae^x + be^{-x}$ is a shifted and stretched hyperbolic sine or cosine function.

6.8 Indeterminate Forms and l'Hospital's Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves *near* 1. In particular, we would like to know the value of the limit

$$\boxed{1} \qquad \lim_{x\to 1} \frac{\ln x}{x - 1}$$

In computing this limit we can't apply Law 5 of limits (the limit of a quotient is the quotient of the limits, see Section 1.6) because the limit of the denominator is 0. In fact, although the limit in (1) exists, its value is not obvious because both numerator and denominator approach 0 and $\frac{0}{0}$ is not defined.

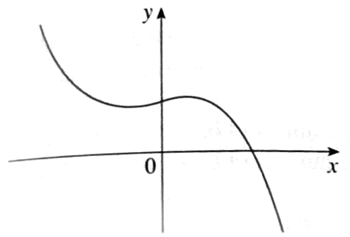
In general, if we have a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

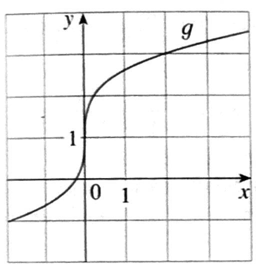
where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type $\frac{0}{0}$**. We met some limits of this type in Chapter 1.

EXERCISES

1. The graph of f is shown. Is f one-to-one? Explain.



2. The graph of g is given.
 (a) Why is g one-to-one?
 (b) Estimate the value of $g^{-1}(2)$.
 (c) Estimate the domain of g^{-1}.
 (d) Sketch the graph of g^{-1}.



3. Suppose f is one-to-one, $f(7) = 3$, and $f'(7) = 8$. Find (a) $f^{-1}(3)$ and (b) $(f^{-1})'(3)$.

4. Find the inverse function of $f(x) = \dfrac{x + 1}{2x + 1}$.

5–9 Sketch a rough graph of the function without using a calculator.

5. $y = 5^x - 1$

6. $y = -e^{-x}$

7. $y = -\ln x$

8. $y = \ln(x + 1)$

9. $y = 2 \arctan x$

10. Let $b > 1$. For large values of x, which of the functions $y = x^b$, $y = b^x$, and $y = \log_b x$ has the largest values and which has the smallest values?

11–12 Find the exact value of each expression.

11. (a) $e^{2 \ln 3}$ (b) $\log_{10} 25 + \log_{10} 4$

12. (a) $\ln e^{\pi}$ (b) $\tan\left(\arcsin \frac{1}{2}\right)$

13–20 Solve the equation for x.

13. $\ln x = \frac{1}{3}$

14. $e^x = \frac{1}{3}$

15. $e^{e^x} = 17$

16. $\ln(1 + e^{-x}) = 3$

17. $\ln(x + 1) + \ln(x - 1) = 1$

18. $\log_5(c^x) = d$

19. $\tan^{-1}x = 1$

20. $\sin x = 0.3$

21–47 Differentiate.

21. $f(t) = t^2 \ln t$

22. $g(t) = \dfrac{e^t}{1 + e^t}$

23. $h(\theta) = e^{\tan 2\theta}$

24. $h(u) = 10^{\sqrt{u}}$

25. $y = \ln|\sec 5x + \tan 5x|$

26. $y = x \cos^{-1}x$

27. $y = x \tan^{-1}(4x)$

28. $y = e^{mx} \cos nx$

29. $y = \ln(\sec^2 x)$

30. $y = \sqrt{t} \ln(t^4)$

31. $y = \dfrac{e^{1/x}}{x^2}$

32. $y = (\arcsin 2x)^2$

33. $y = 3^{x \ln x}$

34. $y = e^{\cos x} + \cos(e^x)$

35. $H(v) = v \tan^{-1}v$

36. $F(z) = \log_{10}(1 + z^2)$

37. $y = x \sinh(x^2)$

38. $y = (\cos x)^x$

39. $y = \ln \sin x - \frac{1}{2} \sin^2 x$

40. $y = \arctan\left(\arcsin \sqrt{x}\right)$

41. $y = \ln\left(\dfrac{1}{x}\right) + \dfrac{1}{\ln x}$

42. $xe^y = y - 1$

43. $y = \ln(\cosh 3x)$

44. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3(3x - 1)^5}$

45. $y = \cosh^{-1}(\sinh x)$

46. $y = x \tanh^{-1}\sqrt{x}$

47. $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

48. Show that

$$\frac{d}{dx}\left(\frac{1}{2}\tan^{-1}x + \frac{1}{4}\ln\frac{(x + 1)^2}{x^2 + 1}\right) = \frac{1}{(1 + x)(1 + x^2)}$$

49–52 Find f' in terms of g'.

49. $f(x) = e^{g(x)}$

50. $f(x) = g(e^x)$

51. $f(x) = \ln|g(x)|$

52. $f(x) = g(\ln x)$

53–54 Find $f^{(n)}(x)$.

53. $f(x) = 2^x$

54. $f(x) = \ln(2x)$

55. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$.

56. Find y' if $y = x + \arctan y$.

57–58 Find an equation of the tangent to the curve at the given point.

57. $y = (2 + x)e^{-x}$, $(0, 2)$

58. $y = x \ln x$, (e, e)

59. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

60. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

61. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

62. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t\to\infty} C(t) = 0$.
(b) Find $C'(t)$, the rate of change of drug concentration in the blood.
(c) When is this rate equal to 0?

Review

63–78 Evaluate the limit.

63. $\lim_{x\to\infty} e^{-3x}$

64. $\lim_{x\to 10^-} \ln(100 - x^2)$

65. $\lim_{x\to 3^-} e^{2/(x-3)}$

66. $\lim_{x\to\infty} \arctan(x^3 - x)$

67. $\lim_{x\to 0^+} \ln(\sinh x)$

68. $\lim_{x\to\infty} e^{-x} \sin x$

69. $\lim_{x\to\infty} \dfrac{1 + 2^x}{1 - 2^x}$

70. $\lim_{x\to\infty} \left(1 + \dfrac{4}{x}\right)^x$

71. $\lim_{x\to 0} \dfrac{e^x - 1}{\tan x}$

72. $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2 + x}$

73. $\lim_{x\to 0} \dfrac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

74. $\lim_{x\to\infty} \dfrac{e^{2x} - e^{-2x}}{\ln(x + 1)}$

75. $\lim_{x\to -\infty} (x^2 - x^3)e^{2x}$

76. $\lim_{x\to 0^+} x^2 \ln x$

77. $\lim_{x\to 1^+} \left(\dfrac{x}{x - 1} - \dfrac{1}{\ln x}\right)$

78. $\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x}$

79–84 Sketch the curve using the guidelines of Section 3.5.

79. $y = e^x \sin x, \quad -\pi \leqslant x \leqslant \pi$

80. $y = \sin^{-1}(1/x)$

81. $y = x \ln x$

82. $y = e^{2x - x^2}$

83. $y = (x - 2)e^{-x}$

84. $y = x + \ln(x^2 + 1)$

85. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

86. Investigate the family of functions $f(x) = cxe^{-cx^2}$. What happens to the maximum and minimum points and the inflection points as c changes? Illustrate your conclusions by graphing several members of the family.

87. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

88. (a) Show that there is exactly one root of the equation $\ln x = 3 - x$ and that it lies between 2 and e.
(b) Find the root of the equation in part (a) correct to four decimal places.

89. A bacteria culture contains 200 cells initially and grows at a rate proportional to its size. After half an hour the population has increased to 360 cells.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 4 hours.
(c) Find the rate of growth after 4 hours.
(d) When will the population reach 10,000?

90. Cobalt-60 has a half-life of 5.24 years.
(a) Find the mass that remains from a 100-mg sample after 20 years.
(b) How long would it take for the mass to decay to 1 mg?

91. The biologist G. F. Gause conducted an experiment in the 1930s with the protozoan *Paramecium* and used the population function

$$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

to model his data, where t was measured in days. Use this model to determine when the population was increasing most rapidly.

92–105 Evaluate the integral.

92. $\displaystyle\int_0^4 \frac{1}{16 + t^2}\, dt$

93. $\displaystyle\int_0^1 ye^{-2y^2}\, dy$

94. $\displaystyle\int_2^5 \frac{dr}{1 + 2r}$

95. $\displaystyle\int_0^1 \frac{e^x}{1 + e^{2x}}\, dx$

96. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx$

97. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

98. $\displaystyle\int \frac{\sin(\ln x)}{x}\, dx$

99. $\displaystyle\int \frac{x + 1}{x^2 + 2x}\, dx$

100. $\displaystyle\int \frac{\csc^2 x}{1 + \cot x}\, dx$

101. $\displaystyle\int \tan x \ln(\cos x)\, dx$

102. $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\, dx$

103. $\displaystyle\int 2^{\tan\theta} \sec^2\theta\, d\theta$

104. $\displaystyle\int \sinh au\, du$

105. $\displaystyle\int \left(\frac{1 - x}{x}\right)^2 dx$

106–108 Use properties of integrals to prove the inequality.

106. $\displaystyle\int_0^1 \sqrt{1 + e^{2x}}\, dx \geqslant e - 1$

107. $\displaystyle\int_0^1 e^x \cos x\, dx \leqslant e - 1$

108. $\displaystyle\int_0^1 x \sin^{-1} x\, dx \leqslant \pi/4$