Math 121, Section(s) 01, 02, Fall 2024

Homework #13

Due **Friday, October 25th** in Gradescope by 11:59 pm ET

Goal: Exploring Convergence of Infinite Series. Focus on Alternating Series Test, and Ratio Test. We will also focus on fluency of training, using multiple tests.

FIRST: Read through and understand the following Examples. Determine whether the given Series Converges Absolutely or Diverges. Justify.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n! \cdot 2^n}$ Try Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\dfrac{(n+1)^{n+1}}{(n+1)!2^{n+1}}}{\dfrac{n^n}{n! \cdot 2^n}}\right| = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{n!}{(n+1)!} \cdot \frac{2^n}{2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n \cancel{(n+1)}}{n^n} \cdot \frac{\cancel{n!}}{\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2} = \lim_{n\to\infty} \cancel{\left(\frac{n+1}{n}\right)^{n}}^{\to e} \cdot \left(\frac{1}{2}\right) = \frac{e}{2} > 1$$

The Original Series $\boxed{\text{Diverges by the Ratio Test}}$.

Ex: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{e^{2n} \cdot\ n! \cdot\ n^n}$ Try Ratio Test:

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\dfrac{(-1)^{n+1}(2(n+1))!}{e^{2(n+1)}\ (n+1)!\ (n+1)^{n+1}}}{\dfrac{(-1)^n (2n)!}{e^{2n}\ n!\ n^n}}\right|$$

$$= \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \cdot \frac{n^n}{(n+1)^{n+1}} \cdot \frac{e^{2n}}{e^{2n+2}} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} \cdot \frac{n^n}{(n+1)^n (n+1)} \cdot \frac{\cancel{e^{2n}}}{\cancel{e^{2n}} e^2} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{(2\cancel{(n+1)})(2n+1)}{\cancel{(n+1)}(n+1)} \cdot \cancel{\left(\frac{n}{n+1}\right)^{n}}^{\to \frac{1}{e}} \cdot \frac{1}{e^2}$$

$$= \lim_{n\to\infty} \frac{2n+1}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \cdot \left(\frac{2}{e^3}\right) = \lim_{n\to\infty} \frac{2 + \cancel{\frac{1}{n}}^{\to 0}}{1 + \cancel{\frac{1}{n}}^{\to 0}} \cdot \left(\frac{2}{e^3}\right) = 2\left(\frac{2}{e^3}\right) = \frac{4}{e^3} < 1$$

The original series is $\boxed{\text{Absolutely Convergent (A.C.) by the Ratio Test}}$.

Now complete the following HW problems

1. Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+4n+7}$. Use **two** Different methods, namely the Integral Test and the Limit Comparison Test, to prove that this series Diverges. *You can skip checking the Integral Test preconditions here this time. yay!*

2. Determine if the given Alternating series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+1}$ Converges or Diverges.

Determine if the given series is Absolutely Convergent or Divergent.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$ 4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{(2n+1)!}$ 5. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ 7. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}\ 100^n}{n!}$ 8. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

9. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

(a) Show that n^{th} Term Divergence Test is **Inconclusive**.

(b) Show that the Ratio Test is **Inconclusive**.

(c) Show that the series Diverges using the Integral Test. Skip checking the 3 preconditions here. **Note:** This is an example where the terms approach 0 but the series Diverges.

10. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^6}$ is Convergent by using the Limit Comparison Test.

Note that this work will be a sample proof of the fact that *Constant multiple of a Convergent series* **is** *Convergent.*

11. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ Diverges using **two** Different methods.

REGULAR OFFICE HOURS

Sunday 6:00–9:00 pm TAs Natalie/Oscar, SMUDD 207

Monday: 12:00–3:00 pm

6:00–9:00 pm TAs Aaron/Oscar, SMUDD 207

Tuesday: 1:00–4:00 pm

6–7:30 pm TA Gretta, SMUDD 207

Wednesday: 1:00-3:00 pm

7:30–9:00 pm TA Natalie, SMUDD 207

Thursday: none for Professor

extras may be added, TBD weekly

6–9:00 pm TAs Gretta/DJ, SMUDD 207

Friday: 12:00–3:00 pm

6:00–9:00 pm TAs Aaron/DJ, SMUDD 207

Train your Convergence Tests Daily Happy Spring Break!