Natural Exponential Function $y = e^x$ Review

In this handout, we will review the Natural Exponential Function $y = e^x$ by studying its related

1. Function properties
2. Limits
3. Derivatives
4. Integrals

Start by reviewing the General Exponential Function $y = a^x$ with a positive *base* $a > 0$.

Note that the **base** a is a fixed Real number and the **exponent** x is the input variable, free to range in $\mathbb{R}$.

If the exponent x is an integer, then we define $\quad a^x = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{x \text{ copies, multiplied}}$

Example: Simplify $6^2 = 6 \cdot 6 = 36$

Example: Simplify $4^3 = 4 \cdot 4 \cdot 4 = 64$

Example: Simplify $3^5 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{5 \text{ copies}} = 243$

If the exponent x is a rational number $\frac{p}{q}$, then we define $\quad a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} \overset{\text{or}}{=} \left(a^{\frac{1}{q}}\right)^p$

Example: Simplify $8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4 \quad$ or $\quad 8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$

Example: Simplify $(25)^{\frac{3}{2}} = \left(\sqrt{25}\right)^3 = 5^3 = 125 \quad$ or $\quad (25)^{\frac{3}{2}} = \sqrt{(25)^3} = \sqrt{15625} = 125$

We have three possible graphs. The exponential function will be *decreasing* if the base satisfies $0 < a < 1$. It will be *constant* value of 1 if the base $a = 1$, and it will be *increasing* if the base $a > 1$.

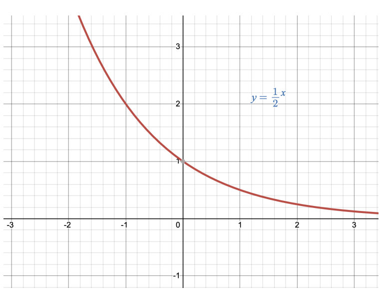

Figure 1: Base $0 < a < 1$

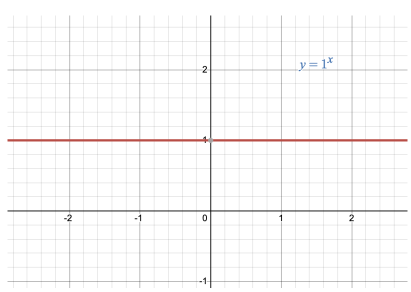

Figure 2: Base $a = 1$

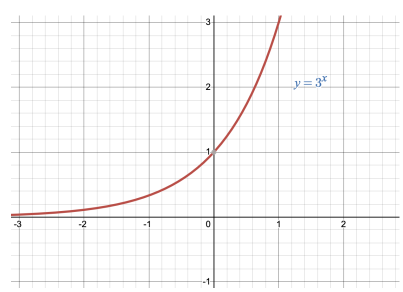

Figure 3: Base $a > 1$

Note that for all of these possible graphs, we have the functions passing through the point $(0, 1)$ since $a^0 = 1$.

IMPORTANT: The exponential functions
- **NEVER** yield the output value $y = 0$
- **NEVER** yield a negative output value

Recall the Limit Definition of the Derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
By studying the long limit definition of the derivative we see that

$$\frac{d}{dx}\left(a^x\right) = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} = \lim_{h\to 0} \frac{a^x\left(a^h-1\right)}{h} = a^x\left[\lim_{h\to 0} \frac{a^h-1}{h}\right]$$

We notice that the derivative of a General Exponential is almost the same as itself, times a complicated limit factor. Here are some samples.

Examples: $\frac{d}{dx}\left(2^x\right) = 2^x(0.693\ldots)$ and $\frac{d}{dx}\left(3^x\right) = 3^x(1.0986\ldots)$

Ideally, this complicated limit multipler would be equal to 1, which motivates the following *special* exponential function.

Definition: The Natural Exponential Function is given by $y = e^x$ where the base e is chosen so that $\frac{d}{dx}e^x = e^x$.

Here $2 < e < 3$ and $e \approx 2.71828\ldots$.

Since the base e is bigger than 1, we know the function is *strictly increasing* with the following important graph

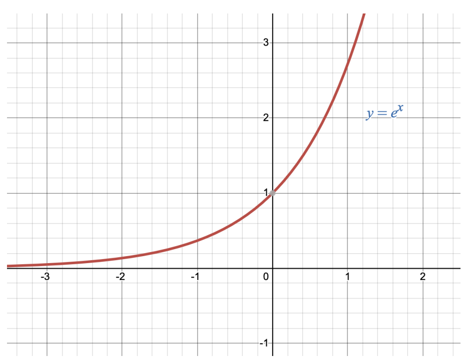

Figure 4: Natural Exponential Function $y = e^x$

Think: The derivative of e^x is itself e^x which is *positive*. Recall, from Calculus I, that **if the derivative is positive, then the function is increasing.** Also, no critical points, why?

Let's study some properties of this Natural Exponential Function $y = e^x$.

Domain$= (-\infty, \infty) = \mathbb{R}$

Recall, the *Domain* of a function is the collection of all possible input values which yield a finite output or the values for which the function is defined. Here, e^x is defined for every possible Real number input.

Range$= (0, \infty) \overset{\text{or}}{=} \{y : y > 0\}$

Recall, the *Range* of a function is the collection of all possible output values for a given function. Here e^x only yields strictly positive output values, that is, never 0 and never negative.

Value(s): $e^0 = 1$

Algebra:

Rule	Tip
$e^a \cdot e^b = e^{a+b}$	with multiplication, if the bases match, then **ADD** the exponents
$\frac{e^a}{e^b} = e^{a-b}$	with division, if the bases match, then **SUBTRACT** the exponents
$(e^a)^b = e^{ab}$	*piggy-back* exponents **MULTIPLY**
$\frac{1}{e^a} = e^{-a}$	move an exponent up or down in fraction, must **CHANGE SIGN**

Limits:

Limit	Tip
$\lim_{x\to\infty} e^x = \infty$	as input values grow uncontrollably large, the e^x output explodes to ∞
$\lim_{x\to-\infty} e^x = 0$	as input grows uncontrollably large *negative*, the e^x output settles to 0

Note: from the graph of $y = e^x$ we see that there is a Horizontal Asymptote at $y = 0$

Derivatives:

Derivative	Tip
$\frac{d}{dx}e^x = e^x$	derivative of natural exponential is exactly itself
$\frac{d}{dx}e^{u(x)} = e^{u(x)} \cdot u'(x)$	**CHAIN RULE** get the original exponential back ... times the derivative of the *inside nested* function

Recall: the Derivative Chain Rule can be written as

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) = \underbrace{f'(g(x))}_{\substack{\text{deriv of outside}\\ \text{leave inside}}} \cdot \underbrace{g'(x)}_{\text{deriv of inside}}}$$

Example: $\dfrac{d}{dx}e^{8x} = e^{8x} \cdot 8 = 8e^{8x}$

Example: $\dfrac{d}{dx}e^{\tan x} = e^{\tan x} \cdot \sec^2 x$

Example: $\dfrac{d}{dx}e^{\sqrt{x}} = e^{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$

Note: If the Exponential is in the denominator, you can sometimes **PREP** the function for the Chain Rule, by moving it to the numerator using our Algebra rule above.

Example: $\dfrac{d}{dx}\left(\dfrac{1}{e^{5x}}\right) \overset{\text{prep}}{=} \dfrac{d}{dx}\left(e^{-5x}\right) = e^{-5x} \cdot (-5) = -5e^{-5x} \overset{\text{or}}{=} \dfrac{-5}{e^{5x}}$

Note: be careful **not** to use the power rule on exponentials.

Helpful WARNING/INCORRECT: $\dfrac{d}{dx}e^{x} \neq xe^{x-1}$

There may be different approaches to compute a more complex derivative. We have many derivative rules. For the next example, involving a product, you can either first FOIL out the algebra and *then* use the Chain Rules on each individual piece, **or** you can first use the Product Rule, and then FOIL out the algebra to simplify. The answers should match. Question: Which method is simpler?

Example: Compute $\dfrac{d}{dx}\left[\left(e^{3x} + e^{-2x}\right)\left(e^{x} - e^{-x}\right)\right]$

METHOD 1 $\rightarrow$ First use Algebra, then use Chain Rule

$$\frac{d}{dx}\left[\left(e^{3x} + e^{-2x}\right)\left(e^{x} - e^{-x}\right)\right] \overset{\text{FOIL}}{=} \frac{d}{dx}\left(e^{4x} - e^{2x} + e^{-x} - e^{-3x}\right)$$

$$\overset{\substack{\text{chain}\\ \text{rule}}}{=} \boxed{4e^{4x} - 2e^{2x} - e^{-x} + 3e^{-3x}}$$

METHOD 2 $\rightarrow$ First use Product Rule, then use Algebra to simplify

$$\frac{d}{dx}\left[\left(e^{3x} + e^{-2x}\right)\left(e^{x} - e^{-x}\right)\right] \overset{\substack{\text{Product}\\ \text{rule}}}{=} \left(e^{3x} + e^{-2x}\right)\left(e^{x} + e^{-x}\right) + \left(e^{x} - e^{-x}\right)\left(3e^{3x} - 2e^{-2x}\right)$$

$$\overset{\text{FOIL}}{=} e^{4x} + e^{2x} + e^{-x} + e^{-3x} + 3e^{4x} - 2e^{-x} - 3e^{2x} + 2e^{-3x}$$

$$\overset{\text{simplify}}{=} \boxed{4e^{4x} - 2e^{2x} - e^{-x} + 3e^{-3x}} \quad \text{Match!}$$

Recall: the Derivative Product Rule can be written as

$$\boxed{\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)}$$

Integrals:

Integral	Tip
$\int e^x \, dx = e^x + C$	antiderivative of natural exponential is exactly itself

Note: be careful **not** to use the power rule on exponentials.

Helpful WARNING/INCORRECT: $\int e^x \, dx \neq \frac{e^{x+1}}{x+1} + C$

At this point in the course, there are only three Methods of Integration:

1. Snap Facts, *we know it*
2. Algebra
3. u-substitution

INDEFINITE Integrals with u-substitution: Always remember to add +C right away, as soon as you compute the Most General Antiderivative. The original variable always reappears when we re-substitute back for u.

Recall: u-substitution is a temporary convenience that hides a nested, meaty chunk of your integrand to first simplify the integral, and second, to match the derivative chunk, all with the overall goal of reversing the Chain Rule.

Example: $$\int \underbrace{\underbrace{e^x}\sin\left(\underbrace{e^x}_{u}\right)\underbrace{dx}}_{du} = \int \sin u \, du = -\cos u + C = \boxed{-\cos\left(e^x\right) + C}$$

$$\boxed{\begin{aligned} u &= e^x \\ du &= e^x \, dx \end{aligned}}$$

Example: $$\int \underbrace{\underbrace{e^x}\left(\underbrace{e^x - 4}_{u}\right)^6\underbrace{dx}}_{du} = \int u^6 \, du = \frac{u^7}{7} + C = \boxed{\frac{\left(e^x - 4\right)^7}{7} + C}$$

$$\boxed{\begin{aligned} u &= e^x - 4 \\ du &= e^x \, dx \end{aligned}}$$

Example: $$\int \frac{\overbrace{\overbrace{e^{5x}}\ \overbrace{dx}}^{du}}{\sqrt{\underbrace{2+e^{5x}}_{u}}} = \frac{1}{5}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{5}\int u^{-\frac{1}{2}}\,du = \frac{1}{5}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{2}{5}\sqrt{u} + C = \boxed{\frac{2}{5}\sqrt{2+e^{5x}} + C}$$

$$\boxed{\begin{aligned} u &= 2+e^{5x} \\ du &= 5e^{5x}\,dx \\ \tfrac{1}{5}du &= e^{5x}\,dx \end{aligned}}$$

Example:
$$\int \frac{5}{e^{6x}\left(8+e^{-6x}\right)^3}\,dx = -\frac{5}{6}\int \frac{1}{u^3}\,du = -\frac{5}{6}\int u^{-3}\,du = -\frac{5}{6}\left(\frac{u^{-2}}{-2}\right) + C$$

$$= \frac{5}{12u^2} + C = \boxed{\frac{5}{12\left(8+e^{-6x}\right)^2} + C}$$

$$\boxed{\begin{aligned} u &= 8+e^{-6x} \\ du &= -6e^{-6x}\,dx \\ -\tfrac{1}{6}du &= \frac{1}{e^{6x}}\,dx \end{aligned}}$$

IMPORTANT: Notice for u-substitution with exponentials, sometimes the exponential *acts as* the u and sometimes the exponential *holds* the u up in its power.

We have a special case for exponential integrals, when the exponential is holding a simpler *constant* times x.

Example: $$\int e^{7x}\,dx = \frac{1}{7}\int e^u\,du = \frac{1}{7}e^u + C = \boxed{\frac{e^{7x}}{7} + C}$$

$$\boxed{\begin{aligned} u &= 7x \\ du &= 7\,dx \\ \tfrac{1}{7}du &= dx \end{aligned}}$$

Example: $$\int e^{-5x}\,dx = -\frac{1}{5}\int e^u\,du = -\frac{1}{5}e^u + C = \boxed{-\frac{e^{-5x}}{5} + C}$$

$$\boxed{\begin{aligned} u &= -5x \\ du &= -5\,dx \\ -\tfrac{1}{5}du &= dx \end{aligned}}$$

"k-Rule" for Natural Exponential : Note, these examples can be generalized to a snap-fact, which we will refer to as the Exponential "k-Rule", so we can avoid justifying the u-substitution each time

Integral	Tip
$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$	Exponential "k-Rule" with constant $k \neq 0$

Practice a few quicker examples, using the snap-fact k-Rule

Example: $\displaystyle\int e^{8x}\,dx = \frac{e^{8x}}{8} + C$ applying the k-Rule

Example: $\displaystyle\int \frac{1}{e^{3x}}\,dx = \int e^{-3x}\,dx = \frac{e^{-3x}}{-3} + C$ applying the k-Rule

INDEFINITE Integrals with FOIL Algebra: examples involving Algebra to FOIL out a complicated function before integrating each piece (using new k-rule).

Example:

$$\int\left(e^x - \frac{1}{e^x}\right)^2 dx = \int \left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right)\,dx \overset{\text{FOIL}}{=} \int e^{2x} - \cancelto{1}{e^0} - \cancelto{1}{e^0} + e^{-2x}\,du$$

$$= \int e^{2x} - 2 + e^{-2x}\,du = \frac{e^{2x}}{2} - 2x + \frac{e^{-2x}}{-2} + C$$

$$= \boxed{\frac{e^{2x}}{2} - 2x - \frac{1}{2e^{2x}} + C}$$

Example:

$$\int\left(e^{2x} - \frac{1}{e^x}\right)\left(e^x - \frac{1}{e^{5x}}\right) dx = \int \left(e^{2x} - e^{-x}\right)\left(e^x - e^{-5x}\right)\,dx$$

$$\overset{\text{FOIL}}{=} \int e^{3x} - e^{-3x} - \cancelto{1}{e^0} + e^{-6x}\,du$$

$$= \frac{e^{3x}}{3} - \frac{e^{-3x}}{-3} - x + \frac{e^{-6x}}{-6} + C$$

$$= \boxed{\frac{e^{3x}}{3} + \frac{1}{3e^{3x}} - x - \frac{1}{6e^{6x}} + C}$$

QUESTION: How do we know which method of integration to use, u-substitution or Algebra or both? Here are two examples that model the different options. Generally, try u-sub and if the derivative du does not match, then you can try Algebra FOIL as a (good) back-up plan. Let us study two similar examples with different integration methods . . .

Example: u-substitution works

$$\int \frac{e^{4x}}{(1+e^{4x})^2}\,dx = \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\left(\frac{u^{-1}}{-1}\right) + C$$

$$= -\frac{1}{4u} + C = \boxed{-\frac{1}{4\left(1+e^{4x}\right)} + C}$$

$$\boxed{\begin{aligned} u &= 1+e^{4x} \\ du &= 4e^{4x}\,dx \\ \tfrac{1}{4}du &= e^{4x}dx \end{aligned}}$$

Example: Algebra plus k-Rules works

$$\int \frac{(1+e^{4x})^2}{e^{4x}}\,dx = \int \frac{(1+e^{4x})(1+e^{4x})}{e^{4x}}\,dx = \int \frac{1+2e^{4x}+e^{8x}}{e^{4x}}\,dx$$

$$\overset{\text{split}}{=} \int \frac{1}{e^{4x}} + \frac{2\cancel{e^{4x}}}{\cancel{e^{4x}}} + \frac{e^{8x}}{e^{4x}}\,dx \overset{\text{prep}}{=} \int e^{-4x} + 2 + e^{4x}\,dx$$

$$\overset{k\text{-rules}}{=} \boxed{\frac{e^{-4x}}{-4} + 2x + \frac{e^{4x}}{4} + C}$$

DEFINITE Integrals: Recall, you must change (or temporarily mark) your Limits of integration. The variables and Limits of Integration change *simultaneously.* Once you *switch* your Limits of Integration to u-values, then the original variable never reappears. We will save example for more interesting values coming soon, once we study Logarithms.

Example:

$$\int_0^1 e^x\sqrt{5-e^x}\,dx = -\int_4^{5-e} \sqrt{u}\,du = -\int_4^{5-e} u^{\frac{1}{2}}\,du = -\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_4^{5-e} = -\frac{2}{3}u^{\frac{3}{2}}\Bigg|_4^{5-e}$$

$$= -\frac{2}{3}\left((5-e)^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = \boxed{-\frac{2}{3}\left((5-e)^{\frac{3}{2}} - 8\right)}$$

$$\boxed{\begin{aligned} u &= 5-e^x \\ du &= -e^x\,dx \\ -du &= e^x dx \end{aligned}} \text{ and } \boxed{\begin{aligned} &x=0 \Rightarrow u = 5-e^0 = 5-1 = 4 \\ &x=1 \Rightarrow u = 5-e \end{aligned}}$$