Exam 3 Spring 26 Answer Key

1(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3x+4)^n}{n^3 \cdot 8^n}$ Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} (3x+4)^{n+1}}{(n+1)^3 \cdot 8^{n+1}}}{\frac{\cancel{(-1)^n} (3x+4)^n}{n^3 \cdot 8^n}} \right| = \lim_{n\to\infty} \left|\frac{(3x+4)^{n+1}}{(3x+4)^n}\right| \cdot \frac{n^3}{(n+1)^3} \cdot \frac{8^n}{8^{n+1}} = \frac{|3x+4|}{8} < 1$$

$|3x+4|$ $\quad \left(\frac{n}{n+1}\right)^3 \to 1 \quad$ $8^{n+1} = 8^n \cdot 8$

Converges by Ratio Test when

$$\frac{|3x+4|}{8} < 1 \Rightarrow |3x+4| < 8 \Rightarrow -8 < 3x+4 < 8 \Rightarrow -12 < 3x < 4 \Rightarrow -4 < x < \frac{4}{3}$$

$-4 \quad -4 \quad -4$ $\qquad \frac{}{3} \quad \frac{}{3} \quad \frac{}{3}$

Manually Check Convergence at Endpoints

Take $x = -4$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [3(-4)+4]^n}{n^3 \cdot 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-8)^n}{n^3 \cdot 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n \cancel{8^n}}{n^3 \cdot \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Even $(-1)^{2n} \to 1$

Convergent p-Series $p = 3 > 1$

Take $x = \frac{4}{3}$ Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n [3(\frac{4}{3})+4]^n}{n^3 \cdot 8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{8^n}}{n^3 \cdot \cancel{8^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

AS $\rightarrow$ $\displaystyle\sum_{n=1}^{n} \frac{1}{n^3}$ Convergent p-Series $p = 3 > 1$

ACT $\Downarrow$

Original Series Converges by ACT

OR

1. Isolate $b_n = \frac{1}{n^3} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^3} = 0$ $\quad (1/\infty \to 0)$
3. Terms Decreasing

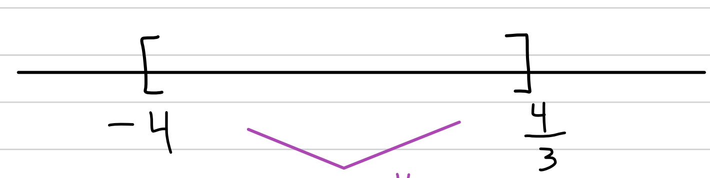
$b_{n+1} = \frac{1}{(n+1)^3} \leq \frac{1}{n^3} = b_n$

Series Converges by AST

only need to show one of AST or ACT

Finally, Interval of Convergence $\boxed{I = \left[-4, \frac{4}{3}\right]}$

Radius of Convergence $\boxed{R = \frac{8}{3}}$



Length $\frac{16}{3}$ $\hookrightarrow$ Half $= \frac{8}{3}$

Total Length $\quad \frac{4}{3} - (-4) = \frac{4}{3} + \frac{12}{3} = \frac{16}{3}$

1(b) $\cos x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{\cancel{(-1)^{n+1}} x^{2(n+1)}}{(2(n+1))!}}{\frac{\cancel{(-1)^n} x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{x^{2n}}\right| \frac{\cancel{(2n)!}}{(2n+2)!} = \lim_{n\to\infty} \frac{|x|^2}{\underbrace{(2n+2)(2n+1)}_{\infty}} \to 0$$

$(2n+2)(2n+1)\cancel{(2n)!}$

$= 0 < 1$ always

Converges by Ratio Test for all x

Finally, $I = (-\infty, \infty)$

$R = \infty$

2(a) $\frac{d}{dx}\left(8x^4 \overset{R=\infty}{\sin(8x)}\right) = \frac{d}{dx}\left(8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (8x)^{2n+1}}{(2n+1)!}\right) = \frac{d}{dx}\, 8x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+1} x^{2n+1}}{(2n+1)!}$

$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2} x^{2n+5}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 8^{2n+2} (2n+5) x^{2n+4}}{(2n+1)!}$

$R = \infty$ STILL After Differentiation

Recall:

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

2(b) $\int \ln(1+9x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (9x^2)^{n+1}}{n+1}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n 9^{n+1} x^{2n+2}}{n+1}\,dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n 9^{n+1} x^{2n+3}}{(n+1)(2n+3)} + C$

Need $|9x^2| < 1 \Rightarrow |x|^2 < \frac{1}{9} \Rightarrow |x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$ before Integration

$\Rightarrow R = \frac{1}{3}$ STILL After Integration

Recall:

$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$

3. Estimate $\int_0^1 x^3 e^{-x^2}\,dx$

$$\int_0^1 x^3 e^{-x^2}\,dx = \int_0^1 x^3 \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx = \int x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{n!\,(2n+4)} \Bigg|_0^1$$

Make Sure to Connect Solution Throughout

$$= \underset{n=0}{\frac{x^4}{1\cdot 4}} - \underset{n=1}{\frac{x^6}{1\cdot 6}} + \underset{n=2}{\frac{x^8}{2!\cdot 8}} - \underset{n=3}{\frac{x^{10}}{3!\cdot(10)}} + \dots \Bigg|_0^1$$

$$= \frac{1}{4} - \frac{1}{6} + \frac{1}{16} - \frac{1}{60} + \dots - \cancel{(0-0+0-\dots)}^{\,0}$$

$$\approx \frac{1}{4} - \frac{1}{6} + \frac{1}{16} = \frac{12}{48} - \frac{8}{48} + \frac{3}{48} = \frac{7}{48}$$ Estimate

Using the Alternating Series Estimation Theorem (A.S.E.T) we can Estimate the Full Sum using only the first three terms with error at most $\frac{1}{60} < \frac{1}{50}$ as desired.

4 a. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n} \cdot \frac{\pi}{3}}{(2n+1)! \cdot \frac{\pi}{3}} = \frac{3}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{3}\right)^{2n+1}}{(2n+1)!}$$

Flip

Recall:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \frac{3}{\pi} \cancelto{\frac{\sqrt{3}}{2}}{\sin\left(\frac{\pi}{3}\right)} = \frac{3\sqrt{3}}{2\pi}$$

4b. $$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\right)$$

$$= -\ln(1+1) = -\ln 2$$

Recall:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

warning: $\ln(1+(-1)) = \ln 0$ undefined won't work

4c. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^n (2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{2^{2n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n}}{(2n)!} = \pi \cancel{\cos}^{0}\left(\frac{\pi}{2}\right) = \pi \cdot 0 = 0$

Recall:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Don't Drop

4d $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} (\ln 9)^n}{2^n \cdot n!} = - \sum_{n=0}^{\infty} \frac{(-1)^n (\ln 9)^n}{2^n n!} = - \sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 9}{2}\right)^n}{n!} = -e^{-\frac{\ln 9}{2}}$

Recall:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= -e^{-\frac{1}{2}\ln 9} = -\cancel{e}^{\cancel{\ln}\left(9^{-\frac{1}{2}}\right)} = -\frac{1}{\sqrt{9}} = -\frac{1}{3}$$

$\frac{1}{9^{1/2}}$

Extra 1

4e. $4 + 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \dots = 4\left(1 + \underbrace{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots}_{\text{arctan } 1}\right)$

Recall: $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \dots$

$$= 4\left[1 + \cancel{\arctan}^{\frac{\pi}{4}}(1)\right] = 4\left(1 + \frac{\pi}{4}\right) = 4 + \pi$$

OR

4f. $-\frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots = \left(\cancel{\cos \pi}^{-1}\right) - 1 = -1 - 1 = -2$

Recall:

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \longrightarrow \cos \pi = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \dots$

missing 1st term

5. Using Series

$$\lim_{x\to 0} \frac{1 + 4x - e^{4x}}{1 - \cos(3x)} \overset{0/0}{=} \lim_{x\to 0} \frac{\cancel{1} + \cancel{4x} - \left(\cancel{1} + \cancel{(4x)} + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!} + \dots\right)}{\cancel{1} - \left(\cancel{1} - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \dots\right)}$$

$$= \lim_{x\to 0} \frac{-\frac{16x^2}{2!} - \frac{4^3 x^3}{3!} - \frac{4^4 x^4}{4!} - \dots}{+\frac{9x^2}{2!} - \frac{3^4 x^4}{4!} + \frac{3^6 x^6}{6!} - \dots} \quad \frac{1}{x^2}$$

$\frac{1}{x^2}$

Show All Algebra Steps

5 (continued)

$$= \lim_{x\to 0} \frac{-\frac{16}{2} - \cancel{\frac{4^3x}{3!}} - \cancel{\frac{4^4x^4}{4!}} - \dots}{\frac{9}{2} - \cancel{\frac{3^4x^2}{4!} + \frac{3^6x^4}{6!}} - \dots} = \frac{-\cancel{\frac{16}{2}}^{\frac{2}{9}}}{\frac{9}{2}} = \frac{-16}{9}$$

☆ Check answer with Optional L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1+4x-e^{4x}}{1-\cos(3x)} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{4-4e^{4x}}{-\mp 3\sin(3x)} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{-16\cancel{e^{4x}}^{1}}{9\cancel{\cos(3x)}^{1}} = \frac{-16}{9}$$ Match!

$$6.\ \ln(8+x^3) = \int \frac{3x^2}{8+x^3}\,dx = \int 3x^2\left(\frac{1}{8+x^3}\right)dx = \int \frac{3x^2}{8}\left(\frac{1}{1+\frac{x^3}{8}}\right)dx = \int \frac{3x^2}{8}\left(\frac{1}{1-\left(-\frac{x^3}{8}\right)}\right)dx$$

$$= \int \frac{3x^2}{8}\sum_{n=0}^{\infty}\left(\frac{-x^3}{8}\right)^n dx = \int \frac{3x^2}{8}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{8^n}\,dx = \int 3\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+2}}{8^{n+1}}\,dx$$

$$= 3\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+3}}{8^{n+1}(3n+3)} + \cancel{C}^{\ln 8}$$

Expand in Long Form to Solve for $+C$

$$\ln(8+x^3) = 3\left(\frac{x^3}{8\cdot 3} - \frac{x^6}{8^2\cdot 6} + \frac{x^9}{8^3\cdot 9} - \dots\right) + C$$

Always start by writing out terms in x in "Long Expanded" Form

Test the Center $x=0$ into both sides to solve for $+C$

$$\cancel{\ln(8+0)}^{\ln 8} = \cancel{0-0+0-\dots}^{0} + C \Rightarrow C = \ln 8$$

Finally, $\ln(8+x^3) = \boxed{\cancel{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+3}}{8^{n+1}\underbrace{(3n+3)}_{\cancel{3}(n+1)}} + \ln 8}$ OR $= \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+3}}{8^{n+1}(n+1)} + \ln 8$

Can Cancel 3 or Not

Need $\left|\frac{-x^3}{8}\right| < 1 \Rightarrow |x|^3 < 8 \Rightarrow |x| < 2 \Rightarrow R=2$ before Integration

$\Rightarrow$ $R = 2$ STILL After Integration

Recall: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

6 (Continued)

Optional Substitution check $\hookrightarrow$ Requires Log Algebra

$$\ln(8+x^3) = \ln\left(8\left(1+\frac{x^3}{8}\right)\right) = \ln 8 + \ln\left(1+\frac{x^3}{8}\right)$$

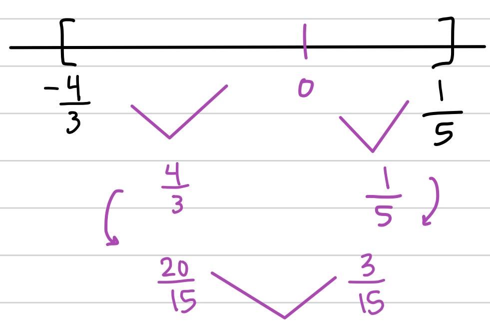
$$= \ln 8 + \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x^3}{8}\right)^{n+1}}{n+1} = \ln 8 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{8^{n+1}(n+1)}$$

Match!

Optional Bonus 1

Need Series with Interval of Convergence $I = \left(-\frac{4}{3}, \frac{1}{5}\right]$

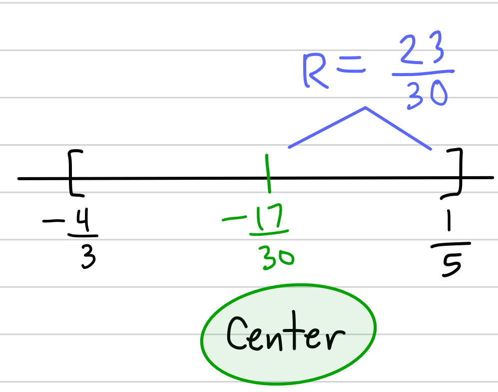
First picture the Interval on the Number Line to find Radius of Convergence



$\frac{23}{15}$ ← Total Length $\Rightarrow$ Radius of Convergence is Half $R = \frac{23}{30}$

Next, find the Center Point using one endpoint and the Radius above

center $a = \underbrace{\frac{1}{5} - \frac{23}{30}}_{\text{right} - \text{Radius}} = \frac{6}{30} - \frac{23}{30} = \underbrace{\frac{-17}{30}}_{\text{center}}$



Reverse Engineer $|x-a| < R$

Therefore, for any x to be contained in I, we need $\left|x - \left(-\frac{17}{30}\right)\right| < \frac{23}{30}$

That is, $\left|x + \frac{17}{30}\right| < \frac{23}{30}$ OR $\frac{30}{23}\left|x + \frac{17}{30}\right| < 1$ OR $\frac{|30x+17|}{23} < 1$

Looks like end of a Ratio Test Computation

Guess. $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n (30x+17)^n}{n \, (23)^n}}$ Run Ratio Test

or Some power piece with $p=1$ (or ≤ 1) to yield convergence at one endpoint but divergence at the other endpoint

think: "conditionally convergent"

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(30x+17)^{n+1}}{n+1 \quad (23)^{n+1}}}{\frac{\cancel{(-1)^n}(30x+17)^n}{n \quad (23)^n}}\right| = \lim_{n\to\infty}\left|\frac{\overbrace{\cancel{(30x+17)^{n+1}}}^{\text{one copy } |30x+17|}}{\cancel{(30x+17)^n}}\right| \cdot \frac{\overset{1}{\cancel{n}}}{\cancel{(n+1)}} \cdot \frac{\cancel{(23)^n}}{\underset{\frac{1}{23}}{\cancel{(23)^{n+1}}}} = \frac{|30x+17|}{23} < 1$$

Converges by Ratio Test when

or when $|30x+17| < 23 \Rightarrow -23 < 30x+17 < 23 \Rightarrow -\frac{4}{3} < x < \frac{1}{5}$ good! Match

$-17 \quad -17 \quad -17$

$\frac{-40 < 30x < 6}{30}$

Manually check Convergence at Endpoints.

Test $x = -\frac{4}{3}$. Series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n\left[\overset{10}{\cancel{30}}\left(-\frac{4}{\cancel{3}}\right)+17\right]^n}{n \cdot (23)^n} = \sum_{n=1}^{\infty}\frac{(-1)^n \overbrace{(-23)^n}^{(-1)^n (23)^n}}{n \cdot (23)^n} = \sum_{n=1}^{\infty}\frac{\overset{1}{\cancel{(-1)^{\overset{\text{even}}{2n}}}} \cdot \cancel{(23)^n}}{n \cdot \cancel{(23)^n}} = \sum_{n=1}^{\infty} \frac{1}{n}$$

$-40+17$

Divergent p-Series $p=1$

Take $x = \frac{1}{5}$. Series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n\left(30\left(\frac{1}{5}\right)+17\right)^n}{n \cdot (23)^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{(23)^n}}{n \cdot \cancel{(23)^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Test using AST

1. Isolate $b_n = \frac{1}{n} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$
3. Terms decreasing

$b_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = b_n$

Converges by A.S.T

So both endpoints included in the Domain; therefore $I = \left(-\frac{4}{3}, \frac{1}{5}\right]$ as desired