Exam 3 Spring 22 Answer Key

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (6x+1)^n}{(6n+1)\cdot 7^n}$

Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(6x+1)^{n+1}}{(6(n+1)+1)7^{n+1}}}{\frac{\cancel{(-1)^n}(6x+1)^n}{(6n+1)\cdot 7^n}}\right| = \lim_{n\to\infty}\left|\frac{(6x+1)^{n+1}}{(6x+1)^n}\right|\cdot\frac{6n+1}{6n+7}\cdot\frac{7^n}{7^{n+1}}$$

$|6x+1|$, 1, $\frac{1}{7}$

$$= \frac{|6x+1|}{7} < 1$$ Converges by Ratio Test when

$$\frac{|6x+1|}{7} < 1 \Rightarrow |6x+1| < 7 \Rightarrow -7 < 6x+1 < 7$$

$-1 \quad -1 \quad -1$

$$-8 < 6x < 6$$

$$-\tfrac{8}{6} < x < 1$$

$-\frac{4}{3}$

Manually Test Convergence at Endpoints:

Take $x = 1$. Series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(6(1)+1)^n}{(6n+1)\cdot 7^n} = \sum_{n=1}^{\infty}\frac{(-1)^n\cdot \cancel{7^n}}{(6n+1)\cdot\cancel{7^n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{6n+1}$

Converges by A.S.T. because

1. $b_n = \dfrac{1}{6n+1} > 0$
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{6n+1} = 0$
3. Terms Decreasing $b_{n+1} = \dfrac{1}{\underbrace{6(n+1)+1}_{6n+7}} \le \dfrac{1}{6n+1} = b_n$

OR, $f(x) = \dfrac{1}{6x+1}$ has

$$f'(x) = \frac{-6}{\underbrace{(6x+1)^2}_{+}} < 0$$

Take $x = -\frac{4}{3}$. Series Becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(\overbrace{6\cdot(-\tfrac{4}{3})}^{-8}+1)^n}{(6n+1)\cdot 7^n} = \sum_{n=1}^{\infty}\frac{\overbrace{(-1)^n(-7)^n}^{(-1)^n 7^n}}{(6n+1)\cdot 7^n} = \sum_{n=1}^{\infty}\frac{\cancel{(-1)^{2n}}}{6n+1}$ Even

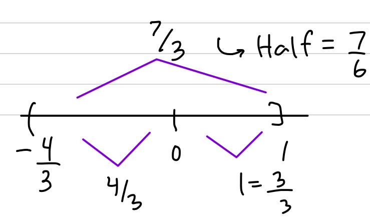
$$= \sum_{n=1}^{\infty}\frac{1}{6n+1} \approx \sum_{n=1}^{\infty}\frac{1}{n}$$ Divergent Harmonic p-Series $p=1$

CT not helpful Bound

LCT $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{6n+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{6n+1} = \frac{1}{6}$ Finite Non-zero

$\Rightarrow$ Series also Diverges by LCT

Finally, $\boxed{I = \left(-\tfrac{4}{3}, 1\right]}$ and $\boxed{R = \tfrac{7}{6}}$



2(a) $\frac{x^2}{4+x} = x^2\left(\frac{1}{4+x}\right) = \frac{x^2}{4}\cdot\frac{1}{1+\frac{x}{4}} = \frac{x^2}{4}\cdot\frac{1}{1-\left(-\frac{x}{4}\right)} = \frac{x^2}{4}\sum_{n=0}^{\infty}\left(-\frac{x}{4}\right)^n$

$$= \frac{x^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{4^n} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+2}}{4^{n+1}}}$$

Need $\left|-\frac{x}{4}\right| = \frac{|x|}{4} < 1 \Rightarrow |x|<4$ $\quad R=4$

2(b) $8x^4\arctan(8x) = 8x^4\sum_{n=0}^{\infty}\frac{(-1)^n(8x)^{2n+1}}{2n+1} = 8x^4\sum_{n=0}^{\infty}\frac{(-1)^n 8^{2n+1}x^{2n+1}}{2n+1}$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 8^{2n+2}x^{2n+5}}{2n+1}}$$

Need $|8x|<1 \Rightarrow |x|<\frac{1}{8}$ $\quad R=\frac{1}{8}$

3(a) $\frac{d}{dx}\left(5x^2e^{-x^3}\right) = \frac{d}{dx}\left(5x^2\sum_{n=0}^{\infty}\frac{(-x^3)^n}{n!}\right) = \frac{d}{dx}\left(5x^2\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{n!}\right)$

$$= \frac{d}{dx}\left(5\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+2}}{n!}\right) = \boxed{5\sum_{n=0}^{\infty}\frac{(-1)^n(3n+2)x^{3n+1}}{n!}}$$

3(b) $\int x^3\sin(8x^4)\,dx = \int x^3\sum_{n=0}^{\infty}\frac{(-1)^n(8x^4)^{2n+1}}{(2n+1)!}dx = \int x^3\sum_{n=0}^{\infty}\frac{(-1)^n 8^{2n+1}x^{8n+4}}{(2n+1)!}dx$

$$= \int\sum_{n=0}^{\infty}\frac{(-1)^n 8^{2n+1}x^{8n+7}}{(2n+1)!}dx = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 8^{2n+1}x^{8n+8}}{(2n+1)!(8n+8)} + C}$$

4. Estimate $\frac{1}{e} = e^{-1}$

Recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ all Full Sums

$$e^{-1} = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \dots$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \dots$$

$$\approx \frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$ Estimate

Using ASET, we can Estimate the Full Sum using only the first two terms with Error at most the Absolute Value of the first Neglected term $\frac{1}{24} < \frac{1}{20}$ as desired

5(a) $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(16)^n (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(4)^{2n} (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n}}{(2n+1)!} \cdot \frac{\frac{\pi}{4}}{\frac{\pi}{4}}$$

flip

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \frac{4}{\pi} \sin\left(\frac{\pi}{4}\right) = \frac{4\sqrt{2}}{2\pi} = \boxed{\frac{2\sqrt{2}}{\pi}}$$

$\frac{\sqrt{2}}{2}$

5(b) $$3 + 3 - \frac{3}{2} + 1 - \frac{3}{4} + \frac{3}{5} - \frac{3}{6} + \frac{3}{7} - \dots$$

$\frac{3}{3}$

extra 3

$$= 3 + 3\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots\right) = 3 + 3\ln(1+1) = 3 + 3\ln 2$$

extra 3

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

extra ⊖, extra π

5(c) $$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} \pi^{2n+1}}{4!\,(2n)!} = -\frac{\pi}{24} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = -\frac{\pi}{24} \cos \pi = \frac{\pi}{24}$$

-1

constant

$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

5(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (\ln 8)^n}{3^{n+1}\, n!} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{\left(-\frac{\ln 8}{3}\right)^n}{n!} = \frac{1}{3} e^{-\frac{\ln 8}{3}}$

extra 3

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad = \frac{1}{3}\cancel{e^{\ln}}\left(8^{-\frac{1}{3}}\right) = \frac{1}{3}\cdot 8^{-1/3} = \frac{1}{3}\cdot\frac{1}{8^{1/3}} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$\sqrt[3]{8}$

5(e) $$-\frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \dots = (\overset{0}{\cancel{\sin\pi}}) - \pi = -\pi$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad \sin\pi = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \dots$$

5(f) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n}} \frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$= \sqrt{3}\, \overset{\frac{\pi}{6}}{\cancel{\arctan\left(\frac{1}{\sqrt{3}}\right)}} = \frac{\sqrt{3}\,\pi}{6}$$

6. $$\ln(1+6x) = \int \frac{6}{1+6x}\,dx = \int 6\left(\frac{1}{1-(-6x)}\right)dx = \int 6\sum_{n=0}^{\infty}(-6x)^n\,dx$$

$$= \int 6\sum_{n=0}^{\infty}(-1)^n 6^n x^n\,dx = \int \sum_{n=0}^{\infty} (-1)^n 6^{n+1} x^n\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 6^{n+1} x^{n+1}}{n+1} + C = 6x - \frac{6^2x^2}{2} + \frac{6^3x^3}{3} - \dots + C$$

Test $x=0$

$$\overset{0}{\cancel{\ln(1+0)}} = 0 - 0 + 0 - \dots + C \quad C = 0$$

Finally, $\ln(1+6x) = \boxed{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 6^{n+1} x^{n+1}}{n+1}}$ Match!

7. $x(t) = (\arctan t) - t \qquad y(t) = 2\sinh^{-1} t$

$$\frac{dx}{dt} = \frac{1}{1+t^2} - 1 \qquad \frac{dy}{dt} = \frac{2}{\sqrt{1+t^2}}$$

$$\text{Arc length} = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} - 1\right)^2 + \left(\frac{2}{\sqrt{1+t^2}}\right)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2}\right)^2 - \frac{2}{1+t^2} + 1 + \frac{4}{1+t^2}}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2}\right)^2 + \frac{2}{1+t^2} + 1}\, dt$$

$$= \int_0^{\sqrt{3}} \sqrt{\left(\frac{1}{1+t^2} + 1\right)^2}\, dt$$

$$= \int_0^{\sqrt{3}} \frac{1}{1+t^2} + 1\, dt$$

$$= \arctan t + t \Big|_0^{\sqrt{3}} = \underbrace{\arctan\sqrt{3}}_{\frac{\pi}{3}} + \sqrt{3} - (\underbrace{\arctan 0}_{0} + 0) = \frac{\pi}{3} + \sqrt{3}$$