Math 121 Midterm Exam #3 April 24, 2015

Answer Key

1. [15 Points] Find the **Interval** and **Radius** of Convergence for the following power series. Analyze carefully and with full justification.

$$\sum_{n=1}^{\infty} \frac{(-1)^n(5x-2)^n}{(n+5)\ 8^n}$$

Use Ratio Test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(5x-2)^{n+1}}{(n+6)8^{n+1}}}{\dfrac{(-1)^n(5x-2)^n}{(n+5)8^n}}\right| = \lim_{n\to\infty}\left|\frac{(5x-2)^{n+1}}{(5x-2)^n}\right|\cdot\left(\frac{n+5}{n+6}\right)\cdot\frac{8^n}{8^{n+1}} = \frac{|5x-2|}{8}$$

The Ratio Test gives convergence for x when $\dfrac{|5x-2|}{8} < 1$ or $|5x-2| < 8$.

That is $-8 < 5x-2 < 8 \Longrightarrow -6 < 5x < 10 \Longrightarrow -\dfrac{6}{5} < x < 2$

Endpoints:

•$x = 2$ The original series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(5(2)-2)^n}{(n+5)\ 8^n} = \sum_{n=1}^{\infty}\frac{(-1)^n 8^n}{(n+5)\ 8^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n+5}$ which is convergent by AST:

1. $b_n = \dfrac{1}{n+5} > 0$

2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n+5} = 0$

3. $b_{n+1} < b_n$ because $b_{n+1} = \dfrac{1}{n+6} < \dfrac{1}{n+5} = b_n$.

OR $f(x) = \dfrac{1}{x+5}$ has derivative $f'(x) = -\dfrac{1}{(x+5)^2} < 0$ so the terms are decreasing.

•$x = -\dfrac{6}{5}$ The original series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\left(5\left(-\frac{6}{5}\right)-2\right)^n}{(n+5)\ 8^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-8)^n}{(n+5)\ 8^n}$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n 8^n}{(n+5)8^n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n+5} = \sum_{n=1}^{\infty}\frac{1}{n+5} \approx \sum_{n=1}^{\infty}\frac{1}{n}$$ the divergent Harmonic Series, $p = 1$.

LCT: $\displaystyle\lim_{n\to\infty}\frac{\frac{1}{n+5}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{n+5} = 1$ which is *finite* and *non-zero*. Therefore, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+5}$ is also divergent by LCT.

Finally, Interval of Convergence $\boxed{I = \left(-\frac{6}{5}, 2\right]}$ with Radius of Convergence $\boxed{R = \frac{8}{5}}$.

2. [12 Points] Find the **MacLaurin series** representation for each of the following functions. **State** the Radius of Convergence for each series. Your answer should be in sigma notation $\sum_{n=0}^{\infty}$.

(a) $f(x) = \dfrac{x}{1+7x} = \dfrac{x}{1-(-7x)} = x\sum_{n=0}^{\infty}(-7x)^n = x\sum_{n=0}^{\infty}(-1)^n 7^n x^n = \boxed{\sum_{n=0}^{\infty}(-1)^n 7^n x^{n+1}}$

Here need $|-7x| < 1$ or $|x| < \dfrac{1}{7}$, so $\boxed{R = \dfrac{1}{7}}$.

(b) $f(x) = x^4 \arctan(4x)$

First, $\arctan x = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{2n+1}}{2n+1}$.

Next, $\arctan(4x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{(4x)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \dfrac{4^{2n+1}x^{2n+1}}{2n+1}$

Finally, $x^4 \arctan(4x) = x^4 \sum_{n=0}^{\infty}(-1)^n \dfrac{4^{2n+1}x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \dfrac{4^{2n+1}x^{2n+5}}{2n+1}$

Here need $|4x| < 1$ or $|x| < \dfrac{1}{4}$, so $\boxed{R = \dfrac{1}{4}}$.

3. [15 Points]

(a) Write the MacLaurin Series representation for $f(x) = x^3 \ln(1+x^3)$.

First $\ln(1+x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+1}}{n+1}$

Second, $\ln(1+x^3) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x^3)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n+3}}{n+1}$

Finally, $x^3 \ln(1+x^3) = x^3 \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n+3}}{n+1} = \boxed{\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n+6}}{n+1}}$

(b) Use the MacLaurin Series representation for $f(x) = x^3 \ln(1+x^3)$ from part (a) to

$$\textbf{Estimate} \int_0^1 x^3 \ln(1+x^3)\, dx \quad \text{with error less than } \frac{1}{30}.$$

Justify in words that your error is indeed less than $\dfrac{1}{30}$.

$$\int_0^1 x^3 \ln(1+x^3)\, dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+6}}{n+1}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+7}}{(n+1)(3n+7)}\Bigg|_0^1$$

$$= \frac{x^7}{1\cdot 7} - \frac{x^{10}}{2\cdot 10} + \frac{x^{13}}{3\cdot 13} - \ldots \Big|_0^1 = \frac{x^7}{7} - \frac{x^{10}}{20} + \frac{x^{13}}{39} - \ldots \Big|_0^1$$

$$= \frac{1}{7} - \frac{1}{20} + \frac{1}{39} - \ldots - (0 - 0 + 0 - \ldots)$$

$$\approx \frac{1}{7} - \frac{1}{20} = \frac{20}{140} - \frac{7}{140} = \boxed{\frac{13}{140}} \quad \leftarrow \quad \textbf{estimate}$$

Using the Alternating Series Estimation Theorem (ASET), we can approximate the actual sum with only the first two terms, and the error from the actual sum will be *at most* the absolute value of the next (first neglected) term, $\frac{1}{39}$. Here $\frac{1}{39} < \frac{1}{30}$ as desired.

4. [18 Points] Find the **sum** for each of the following series.

(a) $\frac{1}{\pi} - \frac{1}{2(\pi)^2} + \frac{1}{3(\pi)^3} - \frac{1}{4(\pi)^4} + \frac{1}{5(\pi)^5} - \frac{1}{6(\pi)^6} + \ldots = \boxed{\ln\left(1 + \frac{1}{\pi}\right)}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(36)^n\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(6)^{2n}\ (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{6}\right)^{2n}}{(2n+1)!} \cdot \left(\frac{\frac{\pi}{6}}{\frac{\pi}{6}}\right) = \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{6}\right)^{2n+1}}{(2n+1)!}$$

$$= \frac{6}{\pi} \sin\left(\frac{\pi}{6}\right) = \frac{6}{\pi}\left(\frac{1}{2}\right) = \boxed{\frac{3}{\pi}}$$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 8)^n}{3^{n+1}\ n!} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n\ (\ln 8)^n}{3^n\ n!} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{\left(\frac{-\ln 8}{3}\right)^n}{n!}$

$$= \frac{1}{3} e^{\left(\frac{-\ln 8}{3}\right)} = \frac{1}{3} e^{\ln\left(8^{-\frac{1}{3}}\right)} = \frac{1}{3}\left(8^{-\frac{1}{3}}\right) = \frac{1}{3}\left(\frac{1}{8^{\frac{1}{3}}}\right) = \frac{1}{3}\left(\frac{1}{2}\right) = \boxed{\frac{1}{6}}$$

$$\text{OR} = \ldots = \frac{1}{3} e^{\left(\frac{-\ln 8}{3}\right)} = \frac{1}{3}\left(\frac{1}{e^{\frac{\ln 8}{3}}}\right) = \frac{1}{3}\left(\frac{1}{e^{\ln\left(8^{\frac{1}{3}}\right)}}\right) = \ldots$$

(d) $\frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3}\,)^3} + \frac{1}{5(\sqrt{3}\,)^5} - \frac{1}{7(\sqrt{3}\,)^7} + \ldots = \arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{2^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{2}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{2}\right) = \boxed{0}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{2^{4n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{(2^2)^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \pi^{2n}}{4^{2n}\ (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n\ \left(\frac{\pi}{4}\right)^{2n}}{(2n)!}$

$$= \cos\left(\frac{\pi}{4}\right) = \boxed{\frac{1}{\sqrt{2}}} = \boxed{\frac{\sqrt{2}}{2}}$$

5. [20 Points] Volumes of Revolution

(a) Consider the region bounded by $y = \arctan x$, $y = 8 - x$, $x = 0$, and $x = 1$. Rotate this region about the horizontal line $\boxed{y = -1}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Washer Method. Sketch the solid, along with one of the approximating washers.

See me for a sketch.

$$V = \int_0^1 \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right]\ dx = \boxed{\pi \int_0^1 (9-x)^2 - (\arctan x + 1)^2\ dx}$$

(b) Consider the region bounded by $y = 1 + e^x$, $y = 4$, and $x = 0$. Rotate this region about the vertical line $\boxed{x = -2}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch. Intersect when $1 + e^x = 4$ or $e^x = 3$ which is when $x = \ln 3$.

$$V = \int_0^{\ln 3} 2\pi \text{ radius height } dx = \boxed{2\pi \int_0^{\ln 3} (x+2)\left(4 - (e^x + 1)\right)\ dx}$$

(c) Consider the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{4}$. Rotate this region about the vertical line $\boxed{x = 3}$. Set-up, **BUT DO NOT EVALUATE!!**, the integral to compute the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch.

$$V = \int_0^{\frac{\pi}{4}} 2\pi \text{ radius height } dx = \boxed{2\pi \int_0^{\frac{\pi}{4}} (3-x)(\cos x - \sin x)\ dx}$$

(d) Consider the region bounded by $y = \ln x$, $y = e^x$, $x = 1$ and $x = 3$. Rotate this region about the $\boxed{y\text{-axis}}$. **<u>COMPUTE</u>** the volume of the resulting solid using the Cylindrical Shells Method. Sketch the solid, along with one of the approximating shells.

See me for a sketch.

$$V = \int_1^3 2\pi \text{ radius height } dx = 2\pi \int_1^3 x\left(e^x - \ln x\right)\ dx$$

$$= 2\pi \left[xe^x - e^x - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\right]\Bigg|_1^3 = 2\pi \left[xe^x - e^x - \frac{x^2}{2}\ln x + \frac{x^2}{4}\right]\Bigg|_1^3 \text{ see below}$$

$$= 2\pi\left[3e^3 - e^3 - \frac{9}{2}\ln 3 + \frac{9}{4} - \left(e - e - \frac{\ln 1}{2} + \frac{1}{4}\right)\right] = 2\pi\left[2e^3 - \frac{9}{2}\ln 3 + \frac{9}{4} - \frac{1}{4}\right]$$

$$= 2\pi\left[2e^3 - \frac{9}{2}\ln 3 + \frac{8}{4}\right] = \boxed{2\pi\left[2e^3 - \frac{9}{2}\ln 3 + 2\right]}$$

See below for the indiviudal IBP's.

$$\int x\ln x\ dx = \left(\frac{x^2}{2}\ln x\right) - \int \frac{x}{2}\ dx = \left(\frac{x^2}{2}\ln x\right) - \frac{x^2}{4} + C$$

I.B.P. $\boxed{\begin{array}{ll} u = \ln x & dv = x dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{array}}$

$$\int xe^x\ dx = xe^x - \int e^x\ dx = xe^x - e^x + C$$

I.B.P. $\boxed{\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}}$

6. [20 Points] Parametric Equations

(a) Consider the Parametric Curve given by $x = \frac{t^3}{3} - \frac{e^{2t}}{2}$ and $y = 2te^t - 2e^t$.

<u>COMPUTE</u> the **arclength** of this parametric curve for $0 \le t \le 1$.

First compute

$$\frac{dx}{dt} = t^2 - e^{2t}$$

$$\frac{dy}{dt} = 2te^t + 2e^t - 2e^t = 2te^t$$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^1 \sqrt{(t^2 - e^{2t})^2 + (2te^t)^2}\ dt$$

$$= \int_0^1 \sqrt{t^4 - 2t^2e^{2t} + e^{4t} + 4t^2e^{2t}}\ dt = \int_0^1 \sqrt{t^4 + 2t^2e^{2t} + e^{4t}}\ dt = \int_0^1 \sqrt{(t^2 + e^{2t})^2}\ dt$$

$$= \int_0^1 t^2 + e^{2t}\ dt = \frac{t^3}{3} + \frac{e^{2t}}{2}\Big|_0^1 = \frac{1}{3} + \frac{e^2}{2} - \left(0 + \frac{e^0}{2}\right)$$

$$= \frac{1}{3} + \frac{e^2}{2} - \frac{1}{2} = \boxed{\frac{e^2}{2} - \frac{1}{6}}$$

(b) Consider a *different* Parametric Curve given by $x = \sin^3 t$ and $y = \cos^3 t$.

COMPUTE the **surface area** obtained by rotating this curve about the x-axis, for $0 \le t \le \frac{\pi}{2}$.

First compute

$$\frac{dx}{dt} = 3\sin^2 t \cos t$$

$$\frac{dy}{dt} = -3\cos^2 t \sin t$$

$$\text{S.A.} = \int_0^{\frac{\pi}{2}} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \cos^3 t \sqrt{\left(3\sin^2 t \cos t\right)^2 + (-3\cos^2 t \sin t)^2}\ dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \cos^3 t \sqrt{9\sin^4 t \cos^2 t + 9\cos^4 t \sin^2 t}\ dt = 2\pi \int_0^{\frac{\pi}{2}} \cos^3 t \sqrt{9\sin^2 t \cos^2 t \left(\sin^2 t + \cos^2 t\right)}\ dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \cos^3 t 3\sin t \cos t\ dt = 6\pi \int_0^{\frac{\pi}{2}} \cos^4 t \sin t\ dt = -\frac{6\pi \cos^5 t}{5}\Big|_0^{\frac{\pi}{2}}$$

$$= -\frac{6\pi}{5}\left(\cos^5\left(\frac{\pi}{2}\right) - \cos 0\right) = -\frac{6\pi}{5}(0-1) = \boxed{\frac{6\pi}{5}}$$

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OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Consider the region bounded by $y = \ln x$, $y = e^x$, $x = 1$, and $x = e$. Rotate this region about the y–axis. Compute the resulting volume using two methods: Cylindrical Shells Method **and** Washer Method.

See me for a sketch.

SHELLS: $V = 2\pi \int_1^e \text{radius} \cdot \text{height}\ dx$

$$V = 2\pi \int_1^e x(e^x - \ln x)\ dx = 2\pi \int_1^e xe^x - x\ln x\ dx = 2\pi \left[xe^x - e^x - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\right]\Big|_1^e$$

$$= 2\pi \left[xe^x - e^x - \frac{x^2}{2}\ln x + \frac{x^2}{4}\right]\Big|_1^e = 2\pi \left[\left(e \cdot e^e - e^e - \frac{e^2}{2}\ln e + \frac{e^2}{4}\right) - \left(e - e - \frac{1}{2}\ln 1 + \frac{1}{4}\right)\right]$$

$$= \boxed{2\pi \left[(e-1)e^e - \frac{e^2}{4} - \frac{1}{4}\right]}$$

See below for the indiviudal IBP's.

$$\int x\ln x\ dx = \left(\frac{x^2}{2}\ln x\right) - \int \frac{x}{2}\ dx = \left(\frac{x^2}{2}\ln x\right) - \frac{x^2}{4} + C$$

I.B.P. $\boxed{\begin{array}{ll} u = \ln x & dv = x dx \\ du = \frac{1}{x} dx & v = \frac{x^2}{2} \end{array}}$

$$\int xe^x \ dx = xe^x - \int e^x \ dx = xe^x - e^x + C$$

I.B.P. $\boxed{\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}}$

WASHERS:

$$V = \int_0^1 \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] \ dy + \int_1^e \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] \ dy$$

$$\ldots + \int_e^{e^e} \pi \left[(\text{outer radius})^2 - (\text{inner radius})^2\right] \ dy$$

$$= \int_0^1 \pi \left[(e^y)^2 - (1)^2\right] \ dy + \int_1^e \pi \left[(e)^2 - (1)^2\right] \ dy + \int_e^{e^e} \pi \left[(e)^2 - (\ln y)^2\right] \ dy$$

$$= \pi \left[\frac{e^{2y}}{2} - y\right] \Bigg|_0^1 + \pi \left[e^2 y - y\right] \Bigg|_1^e + \pi \left[e^2 y - \big(y(\ln y)^2 - 2y \ln y + 2y\big)\right] \Bigg|_e^{e^e}$$

$$= \pi \left[\frac{e^{2y}}{2} - y\right] \Bigg|_0^1 + \pi \left[e^2 y - y\right] \Bigg|_1^e + \pi \left[e^2 y - y(\ln y)^2 + 2y \ln y - 2y\right] \Bigg|_e^{e^e}$$

$$= \pi \left[\left(\frac{e^2}{2} - 1\right) - \left(\frac{e^0}{2} - 0\right)\right] + \pi \left[\big(e^3 - e\big) - \big(e^2 - 1\big)\right] \ldots (\text{continued})$$

$$\ldots + \pi \left[\Big(e^2 e^e - e^e \left(\ln\left(e^e\right)\right)^2 + 2e^e \ln\left(e^e\right) - 2e^e\Big) - \Big(e^3 - e\left(\ln e\right)^2 + 2e \ln e - 2e\Big)\right]$$

$$= \pi \left[\frac{e^2}{2} - \frac{3}{2} + e^3 - e - e^2 + 1 + e^2 e^e - e^2 e^e + 2e \cdot e^e - 2e^e - e^3 + e - 2e + 2e\right]$$

$$= \pi \left[\frac{e^2}{2} - \frac{3}{2} - e^2 + 1 + 2e \cdot e^e - 2e^e\right] = \boxed{\pi \left[2(e-1)e^e - \frac{e^2}{2} - \frac{1}{2}\right]} \text{ MATCH!}$$