Answer Key for Review Packet for Exam 3

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2x+3)^{n+1}}{n+1}}{\frac{(2x+3)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(2x+3)^{n+1}}{(2x+3)^n}\right| \cdot \underbrace{\frac{n}{n+1}}_{\to 1} = |2x+3| < 1$$

Converges by the Ratio Test when

Check: $|2x+3|<1 \Rightarrow -1 < 2x+3 < 1$

$$\begin{aligned} -3 \quad -3 \quad -3 \\ -4 < 2x < -2 \\ -2 < x < -1 \end{aligned}$$

Manually check Convergence at End points

Take $x=-2$. Series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(2(-2)+3)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which is the Convergent Alternating Harmonic Series

Converges by the Alternating Series Test because

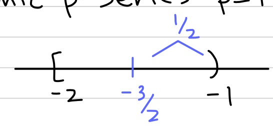
1. Isolate $b_n = \frac{1}{n} > 0$
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$
3. Terms decreasing $b_{n+1} = \frac{1}{n+1} \le \frac{1}{n} = b_n$

Take $x=-1$. Series becomes

$$\sum_{n=1}^{\infty}\frac{(2(-1)+3)^n}{n} = \sum_{n=1}^{\infty}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$ Divergent Harmonic p-Series $p=1$

Finally $I = [-2,-1)$

$R = \frac{1}{2}$



2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n x^n}{n^2\, 4^n}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-3)^{n+1}x^{n+1}}{(n+1)^2\cdot 4^{n+1}}}{\frac{(-3)^n x^n}{n^2\cdot 4^n}}\right| = \lim_{n\to\infty} \underbrace{\frac{3^{n+1}}{3^n}}_{3}\cdot\underbrace{\left|\frac{x^{n+1}}{x^n}\right|}_{|x|}\cdot\underbrace{\frac{n^2}{(n+1)^2}}_{\to 1}\cdot\underbrace{\frac{4^n}{4^{n+1}}}_{1/4} = \frac{3}{4}|x| < 1$$

Converges by Ratio Test when

Check: $\frac{3}{4}|x|<1 \Rightarrow |x| < \frac{4}{3} \Rightarrow -\frac{4}{3} < x < \frac{4}{3}$

Manually Test Convergence at End points

Take $x = -\frac{4}{3}$. Series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n\left(-\frac{4}{3}\right)^n}{n^2\cdot 4^n} = \sum_{n=1}^{\infty}\frac{\overbrace{(-1)^n 3^n (-1)^n}^{(-1)^{2n}\to 1 \text{ Even}} 4^n}{n^2\, 4^n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$

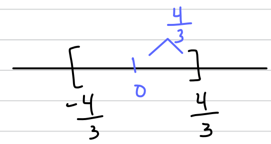
Converges p-Series $p=2>1$

Take $x = \frac{4}{3}$. Series becomes $\sum_{n=1}^{\infty} \frac{(-3)^n \left(\frac{4}{3}\right)^n}{n^2 \cdot 4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{3^n} \cdot \cancel{4^n}/\cancel{3^n}}{n^2 \; \cancel{4^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Converges by the Absolute Convergence Test since its Absolute Series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a Convergent p-Series $p = 2 > 1$. OR can use AST

Finally, $I = \left[-\frac{4}{3}, \frac{4}{3}\right]$

$R = \frac{4}{3}$



3. $\sum_{n=1}^{\infty} \frac{10^n (x+3)^n}{(n+1)^3 \, n!}$ Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{10^{n+1}(x+3)^{n+1}}{(n+2)^3 (n+1)!}}{\frac{10^n (x+3)^n}{(n+1)^3 \cdot n!}} \right| = \lim_{n\to\infty} \frac{\cancel{10^{n+1}}}{\cancel{10^n}}^{10} \cdot \left| \frac{\cancel{(x+3)^{n+1}}}{\cancel{(x+3)^n}} \right|^{|x+3|} \cdot \underbrace{\frac{(n+1)^3}{(n+2)^3}}_{\left(\frac{n+1}{n+2}\right)^3 \to 1} \cdot \frac{\cancel{n!}}{\underbrace{(n+1)!}_{(n+1)\,\cancel{n!}}}$$

$$= \lim_{n\to\infty} \frac{10\,|x+3|}{n+1} \to 0 = 0 < 1$$

Converges by Ratio Test for All Real Numbers x

Finally, $I = (-\infty, \infty)$

$R = \infty$

4. $\sum_{n=1}^{\infty} \frac{2^n (x+1)^n}{5n+1}$ Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(x+1)^{n+1}}{5(n+1)+1}}{\frac{2^n (x+1)^n}{5n+1}} \right| = \lim_{n\to\infty} \frac{\cancel{2^{n+1}}}{\cancel{2^n}}^{2} \left| \frac{\cancel{(x+1)^{n+1}}}{\cancel{(x+1)^n}} \right|^{|x+1|} \cdot \frac{\cancel{5n+1}}{\cancel{5n+6}}^{1} = 2|x+1| < 1$$

Converges by the Ratio Test when

Check: $2|x+1| < 1 \Rightarrow |x+1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x+1 < \frac{1}{2}$

$-1 \quad -1 \quad -1$

$-\frac{3}{2} < x < -\frac{1}{2}$

Check Endpoints

Take $x = -\frac{1}{2}$ Series becomes $\sum_{n=1}^{\infty} \frac{2^n \overbrace{\left(-\frac{1}{2}+1\right)^n}^{\left(\frac{1}{2}\right)^n}}{5n+1} = \sum_{n=1}^{\infty} \frac{\cancel{2^n} \frac{1}{\cancel{2^n}}}{5n+1} = \sum_{n=1}^{\infty} \frac{1}{5n+1} \approx \sum_{n=1}^{\infty} \frac{1}{n}$

Diverges Harmonic p-Series $p = 1$

$\lim_{n\to\infty} \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} = \frac{1}{5}$ Finite Non-Zero

$\Rightarrow$ This Series also Diverges by LCT

Take $x = \frac{-3}{2}$. Series becomes $\sum_{n=1}^{\infty} \frac{2^n \left(-\frac{3}{2}+1\right)^n}{5n+1} = \sum_{n=1}^{\infty} \frac{\cancel{2^n} (-1)^n \cdot \frac{1}{\cancel{2^n}}}{5n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$

$\left(-\frac{1}{2}\right)^n$

Converges by Alternating Series Test

1. Isolate $b_n = \frac{1}{5n+1} > 0$
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{5n+1} = 0$
3. Terms Decreasing

$b_{n+1} = \frac{1}{5n+6} \leq \frac{1}{5n+1} = b_n$

Finally, $I = \left[-\frac{3}{2}, -\frac{1}{2}\right)$

$R = \frac{1}{2}$

$-\frac{3}{2}$ $\quad -1 \quad$ $-\frac{1}{2}$ $\quad \frac{1}{2}$

5. $\sum_{n=0}^{\infty} \frac{(n+2)!\,(x-5)^n}{10^n}$ Ratio Test

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+3)!\,(x-5)^{n+1}}{10^{n+1}}}{\frac{(n+2)!\,(x-5)^n}{10^n}}\right| = \lim_{n\to\infty} \frac{(n+3)!}{(n+2)!}\cdot\left|\frac{(x-5)^{n+1}}{(x-5)^n}\right|\cdot\frac{10^n}{10^{n+1}}$

$(n+3)(n+2)!$ $\quad |x-5|$ $\quad 10$

$= \lim_{n\to\infty} (n+3)\frac{|x-5|}{10} = \infty > 1$

Diverges by the Ratio Test for all x unless $x-5=0$

$x=5$

when $L=0<1$

Finally, $I = \{5\}$

$R = 0$

5

6. $\sum_{n=1}^{\infty} \frac{\sqrt{n}\,(2x-1)^n}{4^n}$ Ratio Test

$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\sqrt{n+1}\,(2x-1)^{n+1}}{4^{n+1}}}{\frac{\sqrt{n}\,(2x-1)^n}{4^n}}\right| = \lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}}\cdot\left|\frac{(2x-1)^{n+1}}{(2x-1)^n}\right|\cdot\frac{4^n}{4^{n+1}} = \frac{|2x-1|}{4} < 1$

$\sqrt{\frac{n+1}{n}} \to 1$ $\quad |2x-1|$ $\quad 4$

Converges by the Ratio Test when

Check: $\frac{|2x-1|}{4} < 1 \Rightarrow |2x-1| < 4 \Rightarrow -4 < 2x-1 < 4$

$+1 \quad +1 \quad +1$

$-3 < 2x < 5$

$-\frac{3}{2} < x < \frac{5}{2}$

Test Endpoints

Take $x = \frac{5}{2}$. Series becomes $\sum_{n=1}^{\infty} \frac{\sqrt{n}\left(\cancel{2}\left(\frac{5}{\cancel{2}}\right)-1\right)^n}{\cancel{4^n}} = \sum_{n=1}^{\infty} \sqrt{n}$

$\cancel{4^n}$

Diverges by n^{th} Term Divergence Test b/c $\lim_{n\to\infty}\sqrt{n} = \infty \neq 0$

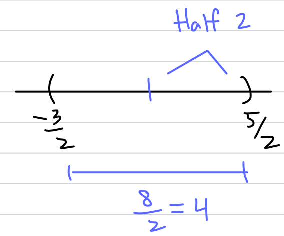
Take $x = -\frac{3}{2}$. Series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}\,\overbrace{\left(\cancel{2}\left(\frac{-3}{\cancel{2}}\right)-1\right)^n}^{(-4)^n}}{4^n} = \sum_{n=1}^{\infty} \frac{\sqrt{n}\,(-1)^n \cancel{4^n}}{\cancel{4^n}} = \sum_{n=1}^{\infty} (-1)^n\sqrt{n}$

Note: $\lim_{n\to\infty} \sqrt{n} = \infty \neq 0 \Rightarrow \lim_{n\to\infty} (-1)^n\sqrt{n} \neq 0$ (DNE, oscillates)

$\Rightarrow$ Series Diverges by nTDT

Finally, $I = \left(-\frac{3}{2}, \frac{5}{2}\right)$

$R = 2$



7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}\,x^{2n+3}}{(2n+3)!}}{\frac{\cancel{(-1)^n}\,x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\overbrace{\left|\frac{x^{2n+3}}{\cancel{x^{2n+1}}}\right|}^{|x^2|} \cdot \frac{\cancel{(2n+1)!}}{\underbrace{(2n+3)!}_{(2n+3)(2n+2)\cancel{(2n+1)!}}} = \lim_{n\to\infty} \frac{|x|^2}{\underbrace{(2n+3)(2n+2)}_{\infty}} \xrightarrow{} 0 = 0 < 1$$

Finally, $I = (-\infty, \infty)$

$R = \infty$

Converges by the Ratio Test for all Real numbers x

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}\,x^{n+1}}{(n+1)^{n+1}}}{\frac{\cancel{(-1)^n}\,x^n}{n^n}}\right| = \lim_{n\to\infty}\overbrace{\left|\frac{x^{n+1}}{\cancel{x^n}}\right|}^{|x|} \cdot \frac{n^n}{\underbrace{(n+1)^{n+1}}_{(n+1)^n\cdot(n+1)}} = \lim_{n\to\infty} \underbrace{\frac{n^n}{(n+1)^n}}_{\frac{1}{e}} \cdot \frac{|x|}{\underbrace{n+1}_{\infty}} \to 0 = 0 < 1$$

Converges by Ratio Test for all x

Finally, $I = (-\infty, \infty)$

$R = \infty$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{(2n)!}{(3n)!}x^n$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2(n+1))!\,x^{n+1}}{(3(n+1))!}}{\frac{(2n)!}{(3n)!}x^n}\right| = \lim_{n\to\infty} \frac{\overbrace{(2n+2)!}^{\overbrace{(2n+2)}^{2\cancel{(n+1)}}(2n+1)\cancel{(2n)!}}}{\cancel{(2n)!}} \cdot \frac{\cancel{(3n)!}}{\underbrace{(3n+3)!}_{\underbrace{(3n+3)}_{3\cancel{(n+1)}}(3n+2)(3n+1)\cancel{(3n)!}}} \cdot \overbrace{\left|\frac{x^{n+1}}{\cancel{x^n}}\right|}^{|x|}$$

$$= \lim_{n\to\infty} \frac{2}{3} \cdot \overbrace{\frac{2n+1}{3n+2}}^{\frac{2}{3}} \cdot \frac{1}{\underbrace{3n+1}_{\infty}} \to 0 \cdot |x| = 0 < 1$$

Converges by Ratio Test for all x

Finally, $I = (-\infty, \infty)$ $R = \infty$

10. $\sum_{n=2}^{\infty} \frac{\ln n}{n^2} \cdot x^n$ Ratio Test

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{\ln(n+1)\, x^{n+1}}{(n+1)^2}}{\frac{\ln n\, x^n}{n^2}} \right| = \lim_{n\to\infty} \cancel{\frac{\ln(n+1)}{\ln n}}^{\,1} \cdot \left| \cancel{\frac{x^{n+1}}{x^n}}^{\,|x|} \right| \cdot \frac{n^2}{(n+1)^2} = |x| < 1$$

★

$$\left(\cancel{\frac{n}{n+1}}^{\,1}\right)^2 \to 1$$

Converges by Ratio Test when

$\Rightarrow -1 < x < 1$

$$\bigstar \quad \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\ln(x+1)}{\ln x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{x}{x+1} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{1} = 1$$

Test Endpoints

Take $x=1$. Series becomes $\sum_{n=2}^{\infty} \frac{\ln n}{n^2} \cdot \cancel{(1)^n}^{\,1} = \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ Converges by Integral Test because

$$\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_2^t \ln x \cdot x^{-2}\,dx = \lim_{t\to\infty} \left. -\frac{\ln x}{x} \right|_2^t + \int_2^t x^{-2}\,dx$$

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^{-2}\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^{-1}}{-1} \end{array}}$$

$$= \lim_{t\to\infty} \left. -\frac{\ln x}{x} \right|_2^t - \left. \frac{1}{x} \right|_2^t$$

$$= \lim_{t\to\infty} \underbrace{\cancel{-\frac{\ln t}{t}}^{\,0}}_{\bigstar\bigstar} + \underbrace{\frac{\ln 2}{4}}_{\text{Finite}} - \cancel{\frac{1}{t}}^{\,0} + \underbrace{\frac{1}{2}}_{\text{Finite}}$$

Integral Converges

$$\bigstar\bigstar \quad \lim_{t\to\infty} \frac{\ln t}{t} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{t\to\infty} \frac{\frac{1}{t}}{1} = \lim_{t\to\infty} \cancel{\frac{1}{t}}^{\,0} = 0$$

Take $x=-1$. Series becomes $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n^2}$ Converges by Absolute Convergence Test b/c its Absolute Series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ Converges by Integral Test, as shown above.

Finally, $I = [-1, 1]$

$R = 1$

Note: Can also use CT on $\sum \frac{\ln n}{n^2}$ using the bound $\ln n \le \sqrt{n}$.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3}(x-1)^n$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}(n+1)!(x-1)^{n+1}}{(n+1)^3}}{\frac{\cancel{(-1)^n}n!(x-1)^n}{n^3}}\right| = \lim_{n\to\infty}\frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}}{\cancel{n!}}\cdot\left|\frac{\cancel{(x-1)^{n+1}}}{\cancel{(x-1)^n}}\right|\cdot\underbrace{\frac{n^3}{(n+1)^3}}_{\left(\frac{n}{n+1}\right)^3\to 1}$$

$$= \lim_{n\to\infty}\underbrace{(n+1)}_{\infty}\cdot|x-1| = \infty > 1$$

Diverges by the Ratio Test unless $x-1=0 \Rightarrow x=1$
$L = 0 < 1$

Finally, $I = \{1\}$
$R = 0$

12. $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^{1/2}}$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{\sqrt{n+1}}}{\frac{x^n}{\sqrt{n}}}\right| = \lim_{n\to\infty}\underbrace{\left|\frac{x^{n+1}}{\cancel{x^n}}\right|}_{|x|}\cdot\underbrace{\frac{\sqrt{n}}{\sqrt{n+1}}}_{\sqrt{\frac{n}{n+1}}\to 1} = |x| < 1$$

Converges by Ratio Test when

$\Rightarrow -1 < x < 1$

Test Endpoints

Take $x=1$. Series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ Diverges p-Series $p = \frac{1}{2} < 1$

Take $x=-1$. Series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ Converges by Alternating Series Test

1. Isolate $b_n = \frac{1}{\sqrt{n}} > 0$
2. $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$
3. Terms Decreasing

$$b_{n+1} = \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} = b_n$$

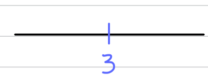
Finally, $I = [-1, 1)$
$R = 1$

13. $\displaystyle\sum_{n=1}^{\infty}(n+4)!\,n^n(x-3)^n$ Ratio Test

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\overbrace{(n+5)!}^{(n+5)\cancel{(n+4)!}}\overbrace{(n+1)^{n+1}}^{(n+1)^n(n+1)}\overbrace{(x-3)^{n+1}}^{|x-3|}}{\cancel{(n+4)!}\,n^n\,\cancel{(x-3)^n}}\right| = \lim_{n\to\infty}\underbrace{(n+5)}_{\infty}\cdot\underbrace{\frac{(n+1)^n}{n^n}}_{e}\cdot\underbrace{(n+1)}_{\infty}|x-3| = \infty > 1$$

Diverges by Ratio Test unless $x-3=0 \Rightarrow x=3$
$L = 0 < 1$

Finally, $I = \{3\}$
$R = 0$



14 $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \dots$

$= 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \dots$

$\approx 1 - \frac{1}{2} + \frac{1}{24} = \frac{24}{24} - \frac{12}{24} + \frac{1}{24} = \frac{13}{24}$ ← Estimate

Using the Alternating Series Estimation Theorem
the Error is at Most $\frac{1}{720} < \frac{1}{100}$ as desired.

15. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$e^{-\frac{1}{3}} = 1 - \frac{1}{3} + \frac{\left(-\frac{1}{3}\right)^2}{2!} + \frac{\left(-\frac{1}{3}\right)^3}{3!} + \frac{\left(-\frac{1}{3}\right)^4}{4!} + \dots$

$\begin{array}{r} 4 \\ 27 \\ \times\ 6 \\ \hline 162 \end{array}$

$= 1 - \frac{1}{3} + \frac{\frac{1}{9}}{\frac{2}{1}} - \frac{\frac{1}{27}}{\frac{6}{1}} + \dots$

$= 1 - \frac{1}{3} + \frac{1}{18} - \frac{1}{162} + \dots$

$\approx 1 - \frac{1}{3} + \frac{1}{18} = \frac{18}{18} - \frac{6}{18} + \frac{1}{18} = \frac{13}{18}$ ← Estimate

Using the Alternating Series Estimation Theorem
the Error is at Most $\frac{1}{162} < \frac{1}{100}$ as desired.

16. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} - \dots$

$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

$\approx 1 - \frac{1}{3} + \frac{1}{5} = \frac{15}{15} - \frac{5}{15} + \frac{3}{15} = \frac{13}{15}$ ← Estimate

Using the Alternating Series Estimation Theorem
the Error is at Most $\frac{1}{7} < \frac{1}{5}$ as desired.

$.20 = \frac{2}{10} = \frac{1}{5}$

17. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$\frac{1}{e} = e^{-1} = 1 + (-1) + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \frac{(-1)^4}{4!} + \frac{(-1)^5}{5!}$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

$$= \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \cdots$$

$$\approx \frac{1}{2} - \frac{1}{6} = \frac{3}{6} - \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \longleftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{24} < \frac{1}{10}$ as desired.

18. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$$

$$= 1 - \frac{1}{6} + \frac{1}{120} - \cdots$$

$$\approx 1 - \frac{1}{6} = \frac{5}{6} \longleftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{120} < \frac{1}{100}$ as desired.

19. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$\frac{1}{\sqrt{e}} = e^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right) + \frac{\left(-\frac{1}{2}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)^3}{3!} + \frac{\left(-\frac{1}{2}\right)^4}{4!} - \cdots$$

$$= 1 - \frac{1}{2} + \frac{\frac{1}{4}}{2} - \frac{\frac{1}{8}}{6} + \frac{\frac{1}{16}}{24} - \cdots$$

$$\begin{array}{r} 24 \\ \times 16 \\ \hline 144 \\ 240 \\ \hline 384 \end{array}$$

$$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384} - \cdots$$

$$\approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} = \frac{48}{48} - \frac{24}{48} + \frac{6}{48} - \frac{1}{48} = \frac{29}{48} \longleftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{384} < \frac{1}{100}$ as desired.

20. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$

$$\sin\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^5}{5!} - \ldots$$

$$= \frac{1}{2} - \frac{\frac{1}{8}}{6} + \frac{\frac{1}{32}}{120} - \ldots$$

$$\begin{array}{r} 120 \\ \times\ 32 \\ \hline 240 \\ 3600 \\ \hline 3840 \end{array}$$

$$= \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} - \ldots$$

$$\approx \frac{1}{2} - \frac{1}{48} = \frac{24-1}{48} = \frac{23}{48} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{3840} < \frac{1}{100}$ as desired.

21. $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots$

$$\arctan\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \ldots$$

$$= \frac{1}{2} - \frac{\frac{1}{8}}{3} + \frac{\frac{1}{32}}{5} - \ldots$$

$$= \frac{1}{2} - \frac{1}{24} + \frac{1}{160} - \ldots$$

$$\approx \frac{1}{2} - \frac{1}{24} = \frac{12}{24} - \frac{1}{24} = \frac{11}{24} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{160} < \frac{1}{100}$ as desired.

22. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - \dfrac{x^6}{6} + \ldots$

$$\ln 2 = \ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

$$\approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

$$= \frac{60}{60} - \frac{30}{60} + \frac{20}{60} - \frac{15}{60} + \frac{12}{60} = \frac{47}{60} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{6} < \frac{1}{5}$ as desired.

23. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\cos\left(\frac{1}{2}\right) = 1 - \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^4}{4!} - \frac{\left(\frac{1}{2}\right)^6}{6!} + \dots$$

$$= 1 - \frac{\frac{1}{4}}{2} + \frac{\frac{1}{16}}{24} - \dots$$

$$= 1 - \frac{1}{8} + \frac{1}{384} - \dots$$

$$\approx 1 - \frac{1}{8} = \frac{7}{8} \leftarrow \text{Estimate}$$

$24 \cdot 16 = 144 + 240 = 384$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{384} < \frac{1}{100}$ as desired.

24. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$$\ln\left(\frac{3}{2}\right) = \ln\left(1+\frac{1}{2}\right) = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \dots$$

$$= \frac{1}{2} - \frac{\frac{1}{4}}{2} + \frac{\frac{1}{8}}{3} - \dots$$

$$= \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \dots$$

$$\approx \frac{1}{2} - \frac{1}{8} = \frac{4}{8} - \frac{1}{8} = \frac{3}{8} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at Most $\frac{1}{24} < \frac{1}{10}$ as desired.

25. $$x^2 e^{-3x^4} = x^2 \sum_{n=0}^{\infty} \frac{(-3x^4)^n}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n x^{4n+2}}{n!} \quad R=\infty$$

$R=\infty$

What will Cancel? Expand Long Form?!

26. $$\frac{1-e^{-x}}{x} = \frac{1}{x}\left(1 - \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) = \frac{1}{x}\left(\cancel{1} - \left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)\right)$$

$R=\infty$

$n=0 \quad n=1 \quad n=2$

$$= \frac{1}{x}\left(-\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n-1}}{n!} \quad R=\infty$$

R=1

27. $$x^4\ln(1+x^3)=x^4\sum_{n=0}^{\infty}\frac{(-1)^n(x^3)^{n+1}}{n+1}=x^4\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+3}}{n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n+7}}{n+1}$$

Need $|x^3|<1 \Rightarrow |x|<1 \Rightarrow R=1$

28. $$\frac{x^6}{1+7x}=x^6\left(\frac{1}{1+7x}\right)=x^6\left(\frac{1}{1-(-7x)}\right)=x^6\sum_{n=0}^{\infty}(-7x)^n$$

Need $|-7x|<1$

$|-7x|=|7x|<1$

$|x|<\frac{1}{7}$ $R=\frac{1}{7}$

$$=x^6\sum_{n=0}^{\infty}(-1)^n 7^n x^n=\sum_{n=0}^{\infty}(-1)^n 7^n x^{n+6}$$

29. $$x\arctan(2x)=x\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n+1}}{2n+1}=x\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}x^{2n+1}}{2n+1}$$

R=1

$$=\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}x^{2n+2}}{2n+1}$$

Need $|2x|<1 \Rightarrow |x|<\frac{1}{2}$ $R=\frac{1}{2}$

30. $$\frac{d}{dx}x^5\sin(x^3)=\frac{d}{dx}x^5\sum_{n=0}^{\infty}\frac{(-1)^n(x^3)^{2n+1}}{(2n+1)!}=\frac{d}{dx}x^5\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+3}}{(2n+1)!}$$

R=∞

$$=\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{6n+8}}{(2n+1)!}=\sum_{n=0}^{\infty}\frac{(-1)^n(6n+8)x^{6n+7}}{(2n+1)!}$$

$R=\infty$ STILL (after differentiation)

31. $$\int 3x e^{-3x^7}dx=\int 3x\sum_{n=0}^{\infty}\frac{(-3x^7)^n}{n!}dx=\int 3x\sum_{n=0}^{\infty}\frac{(-1)^n 3^n x^{7n}}{n!}dx$$

R=∞

$$=\int\sum_{n=0}^{\infty}\frac{(-1)^n 3^{n+1}x^{7n+1}}{n!}dx=\sum_{n=0}^{\infty}\frac{(-1)^n 3^{n+1}}{n!}\cdot\frac{x^{7n+2}}{(7n+2)}+C$$

$R=\infty$ STILL (after integration)

32. $\frac{d}{dx} x^4 \ln(1+8x) = \frac{d}{dx} x^4 \sum_{n=0}^{\infty} \frac{(-1)^n (8x)^{n+1}}{n+1} = \frac{d}{dx} x^4 \sum_{n=0}^{\infty} \frac{(-1)^n 8^{n+1} x^{n+1}}{n+1}$

$R=1$

Need $|8x|<1$

$|x|<\frac{1}{8}$

$R=\frac{1}{8}$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n 8^{n+1} x^{n+5}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 8^{n+1} (n+5) x^{n+4}}{n+1}$$

$R=\frac{1}{8}$ STILL (after differentiation)

33. $\int 6x^3 \cos(6x^2)\,dx = \int 6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (6x^2)^{2n}}{(2n)!}\,dx = \int 6x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n} x^{4n}}{(2n)!}\,dx$

$R=\infty$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+1} x^{4n+3}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n 6^{2n+1} x^{4n+4}}{(2n)!\,(4n+4)} + C$$

$R=\infty$ STILL (after integration)

34. $\frac{1}{(1+7x)^2} = \frac{d}{dx}\left(\frac{-1}{7(1+7x)}\right) = \frac{d}{dx}\left(\frac{-1}{7(1-(-7x))}\right) = \frac{d}{dx}\left(-\frac{1}{7}\sum_{n=0}^{\infty}(-7x)^n\right)$

P.S.

Need $|-7x|<1$

$|7x|<1$

$|x|<\frac{1}{7}$

$R=\frac{1}{7}$

$$= \frac{d}{dx}\left(-\frac{1}{7}\sum_{n=0}^{\infty}(-1)^n 7^n x^n\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^{n+1} 7^{n-1} x^n\right)$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1} 7^{n-1}\, n\, x^{n-1}$$

$R=\frac{1}{7}$ STILL (after differentiation)

35. $\int_0^1 x^2 \cos(x^3)\,dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!}\,dx = \int_0^1 x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}\,dx$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n)!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n)!\,(6n+3)} \Bigg|_0^1$$

35. continued...

$$= \left.\frac{x^3}{1\cdot 3} - \frac{x^9}{2!\cdot 9} + \frac{x^{15}}{4!\cdot(15)} - \cdots\right|_0^1$$

$$\begin{array}{r} 24 \\ 15 \\ \hline 120 \\ 240 \\ \hline 360 \end{array}$$

$$= \frac{1}{3} - \frac{1}{18} + \frac{1}{360} - \cdots - \cancel{(0 - 0 + 0 - \cdots)}^{\,0}$$

$$\approx \frac{1}{3} - \frac{1}{18} = \frac{6}{18} - \frac{1}{18} = \frac{5}{18} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at most $\frac{1}{360} < \frac{1}{50}$ as desired.

36. $$\int_0^{\frac{1}{2}} x \arctan x \, dx = \int_0^{\frac{1}{2}} x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1} dx$$

$$= \left.\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)(2n+3)}\right|_0^{\frac{1}{2}}$$

$$= \left.\frac{x^3}{1\cdot 3} - \frac{x^5}{3\cdot 5} + \frac{x^7}{5\cdot 7} - \cdots\right|_0^{\frac{1}{2}}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^5}{15} + \cdots - \cancel{(0 - 0 + 0 - \cdots)}^{\,0}$$

$$= \frac{\frac{1}{8}}{\frac{3}{1}} - \frac{\frac{1}{32}}{\frac{15}{1}}$$

$$\begin{array}{r} 32 \\ 15 \\ \hline 160 \\ 320 \\ \hline 480 \end{array}$$

$$= \frac{1}{24} - \frac{1}{480} + \cdots$$

$$\approx \frac{1}{24} \leftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at most $\frac{1}{480} < \frac{1}{100}$ as desired.

0.01

$$37.\quad \int_0^1 \sin(x^2)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{4n+3}}{(2n+1)!\,(4n+3)} \Bigg|_0^1 = \overset{n=0}{\frac{x^3}{1\cdot 3}} - \overset{n=1}{\frac{x^7}{3!\cdot 7}} + \dots \Bigg|_0^1$$

$$= \frac{1}{3} - \frac{1}{42} + \dots - \left(\cancelto{0}{0-0+0-\dots}\right)$$

$$\approx \frac{1}{3} \longleftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at most $\frac{1}{42} < \underset{0.1}{\frac{1}{10}}$ as desired.

$$38.\quad \int_0^{\frac{1}{2}} e^{-x^3}\,dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!}\,dx = \int_0^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \, x^{3n+1}}{n!\,(3n+1)} \Bigg|_0^{\frac{1}{2}} = \overset{n=0}{\frac{x^1}{1\cdot 1}} - \overset{n=1}{\frac{x^4}{1\cdot 4}} + \frac{x^7}{2!\cdot 7} - \dots \Bigg|_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^4}{4} + \frac{\left(\frac{1}{2}\right)^7}{14} - \dots - \left(\cancelto{0}{0-0+0-\dots}\right)$$

$$= \frac{1}{2} - \frac{\frac{1}{16}}{\frac{4}{1}} + \frac{\frac{1}{128}}{\frac{14}{1}} - \dots \qquad\qquad \begin{array}{r} {\scriptstyle 1\ 3} \\ 128 \\ \times\ 14 \\ \hline 512 \\ 1280 \\ \hline 1792 \end{array}$$

$$= \frac{1}{2} - \frac{1}{64} + \frac{1}{1792} - \dots$$

$$\approx \frac{1}{2} - \frac{1}{64} = \frac{32}{64} - \frac{1}{64} = \frac{31}{64} \longleftarrow \text{Estimate}$$

Using the Alternating Series Estimation Theorem the Error is at most $\frac{1}{1792} < \underset{0.1}{\frac{1}{100}}$ as desired.

$n=0 \qquad n=1 \qquad n=2$

39. $$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{3^n} = \frac{2^2}{1} - \frac{2^3}{3^1} + \frac{2^4}{3^2} - \ldots$$

Geometric

$a = 4$

$r = -\frac{2}{3}$

$$\text{Sum} = \frac{a}{1-r} = \frac{4}{1-\left(-\frac{2}{3}\right)} = \frac{4}{\frac{5}{3}} \xrightarrow{\frac{3}{5}} = \frac{12}{5}$$

OR// Muscle to $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ form

Same!

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+2}}{3^n} = 2^2 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^n} = 4 \sum_{n=0}^{\infty} \left(\frac{-2}{3}\right)^n = 4\left(\frac{1}{1-\left(-\frac{2}{3}\right)}\right) = 4 \cdot \frac{1}{5/3} \xrightarrow{3/5} = \frac{12}{5}$$

40. $$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots = e^1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

41. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \cos \pi = -1$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

42. $$\sum_{n=0}^{\infty} \frac{(-1)^n 49^n \pi^{2n}}{4^n (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 7^{2n} \pi^{2n}}{2^{2n}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{7\pi}{2}\right)^{2n} \cdot \frac{7\pi}{2}}{(2n+1)! \cdot \frac{7\pi}{2}}$$

Flip to the front

$$= \frac{2}{7\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{7\pi}{2}\right)^{2n+1}}{(2n+1)!} = \frac{2}{7\pi} \sin\left(\frac{7\pi}{2}\right) \xrightarrow{-1} = \frac{-2}{7\pi}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

43. $$\sum_{n=0}^{\infty} \frac{(-9)^n \pi^{2n+1}}{4^n (2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} \pi^{2n}}{2^{2n}(2n)!} = \pi \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{3\pi}{2}\right)^{2n}}{(2n)!}$$

extra

$$= \pi \cos\left(\frac{3\pi}{2}\right) \xrightarrow{0} = \pi \cdot 0 = 0$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

44. $$\sum_{n=0}^{\infty}\frac{(-\pi^2)^n}{36^n(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\,\pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

45. $$\sum_{n=0}^{\infty}\frac{x^{7n+1}}{n!} = x\sum_{n=0}^{\infty}\frac{x^{7n}}{n!} = x\sum_{n=0}^{\infty}\frac{(x^7)^n}{n!} = x\,e^{x^7}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

46. $$-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots = \left(\overset{\frac{\pi}{4}}{\cancel{\arctan 1}}\right)-1 = \frac{\pi}{4}-1 = \frac{\pi-4}{4}$$

OR

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots \qquad \arctan 1 = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$$

missing

47. $$1-\frac{1}{2}+\frac{1}{2^2\cdot 2!}-\frac{1}{2^3\cdot 3!}+\frac{1}{2^4\cdot 4!}-\dots$$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

not alternating, so "absorb" minus

$$= 1-\frac{1}{2}+\frac{\left(\frac{1}{2}\right)^2}{2!}-\frac{\left(\frac{1}{2}\right)^3}{3!}+\frac{\left(\frac{1}{2}\right)^4}{4!}-\dots$$

$$= 1+\left(-\frac{1}{2}\right)+\frac{\left(-\frac{1}{2}\right)^2}{2!}+\frac{\left(-\frac{1}{2}\right)^3}{3!}+\frac{\left(-\frac{1}{2}\right)^4}{4!}+\dots = e^{-\frac{1}{2}}$$

48. $$\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)^{n+1}}{n+1} = \ln\left(1+\frac{1}{2}\right) = \ln\left(\frac{3}{2}\right)$$

$$\ln(1+x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$$

49. $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\,3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(\sqrt{3})^{2n+1}}$$

$$= \sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1} = \sqrt{3}\,\overset{\frac{\pi}{6}}{\cancel{\arctan\left(\frac{1}{\sqrt{3}}\right)}} = \frac{\sqrt{3}\,\pi}{6}$$

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$$

50. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1} = \left(\cancel{\arctan 1}^{\,\frac{\pi}{4}}\right) - \Big(\underbrace{1 - \tfrac{1}{3}}_{\text{missing } n=0,1 \text{ terms}}\Big)^{\frac{2}{3}} = \frac{\pi}{4} - \frac{2}{3}$

$n=2$ ← Check!

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\arctan 1 = \underbrace{1 - \tfrac{1}{3}}_{\substack{n=0 \;\; n=1 \\ \text{missing}}} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots$$

51. $\displaystyle\lim_{x\to 0} \frac{\sin(3x) - 3x}{x - \arctan x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{3\cos(3x) - 3}{1 - \frac{1}{1+x^2}} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{-9\sin(3x)}{\frac{2x}{(1+x^2)^2}}$ $0/0$

$-(1+x^2)^{-1}$

$\hookrightarrow +(1+x^2)^{-2}(2x)$

$$\underset{L'H}{=} \lim_{x\to 0} \frac{-27\cos(3x)^{\to 0}}{\frac{(1+x^2)^2 \cdot 2 - 2x \cdot 2(1+x^2)(2x)}{(1+x^2)^4}} = -\frac{27}{2}$$

Messy Quotient Rule

$\hookrightarrow$ Makes Series Nicer

Now Series

$$\lim_{x\to 0} \frac{\sin(3x) - 3x}{x - \arctan x} = \lim_{x\to 0} \frac{\cancel{3x} - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \ldots - \cancel{3x}}{\cancel{x} - \left(\cancel{x} - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)}$$

Match!

$$= \lim_{x\to 0} \frac{-\frac{27x^3}{3!} + \frac{3^5 x^5}{5!} - \ldots}{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \ldots} \quad \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$$

$$= \lim_{x\to 0} \frac{-\frac{27}{6} + \frac{3^5 x^2}{5!} - \ldots}{\frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \ldots} = \frac{-\frac{27}{\cancel{6}_{\,2}}}{\frac{1}{3}}\,\cancel{3} = -\frac{27}{2}$$

52. $\displaystyle\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+3x) - 3x} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{xe^x + e^x - \frac{1}{1+x^2}}{\frac{3}{1+3x} - 3} \overset{0/0}{\underset{L'H}{=}} \lim_{x\to 0} \frac{xe^x + e^x + e^x + \frac{2x}{(1+x^2)^2}}{\frac{-9}{(1+3x)^2}}$

$-(1+x^2)^{-1} \to +(1+x^2)^{-2}(2x)$

$3(1+3x)^{-1} \hookrightarrow -3(1+3x)^{-2}(3)$

$$= -\frac{2}{9}$$

52 continued

Now Series

$$\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+3x) - 3x} = \lim_{x\to 0} \frac{x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\dots\right)}{\cancel{3x}-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}-\dots \quad -\cancel{3x}}$$

$$= \lim_{x\to 0} \frac{\cancel{x}+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\dots-\cancel{x}+\frac{x^3}{3}-\frac{x^5}{5}+\dots}{-\frac{9x^2}{2}+\frac{27x^3}{3}-\dots}$$

$$= \lim_{x\to 0} \frac{x^2+\frac{x^3}{2!}+\frac{x^3}{3}+\frac{x^4}{3!}-\dots}{-\frac{9x^2}{2}+\frac{27x^3}{3}} \quad \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0} \frac{1+\cancel{\frac{x}{2}+\frac{x}{3}+\frac{x^2}{3!}-\dots} \to 0}{-\frac{9}{2}+\cancel{\frac{27x}{3}-\dots} \to 0}$$

$$= \frac{1}{-\frac{9}{2}} = \frac{-2}{9}$$

Match!

53. Study Related Series

$\sum_{n=0}^{\infty} \frac{6^n}{n!}$ Converges by Ratio Test because

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{6^{n+1}}{(n+1)!}}{\frac{6^n}{n!}}\right| = \lim_{n\to\infty} \frac{\cancel{6^{n+1}}^{\,6}}{\cancel{6^n}} \cdot \frac{\cancel{n!}}{(n+1)!} = \lim_{n\to\infty} \frac{6}{\cancel{n+1}} = 0 < 1$$

$(n+1)\cancel{n!}$; $\to 0$; $\to \infty$

Therefore, $\lim_{n\to\infty} \frac{6^n}{n!} = 0$ because otherwise, if $\lim_{n\to\infty} \frac{6^n}{n!} \neq 0$ then the

Series $\sum \frac{6^n}{n!}$ would Diverge by the n^{th} Term Divergence Test,

which would contradict our proof of Convergence above.

54. Study Related Series

$\sum_{n=0}^{\infty} \frac{n^n \cdot n!}{(3n)!}$ Converges by the Ratio Test because

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}(n+1)!}{(3(n+1))!}}{\frac{n^n \, n!}{(3n)!}}\right| = \lim_{n\to\infty} \frac{\overbrace{(n+1)^{n+1}}^{(n+1)^n(n+1)}}{n^n} \cdot \frac{\overbrace{(n+1)!}^{(n+1)\cancel{n!}}}{\cancel{n!}} \cdot \frac{\cancel{(3n)!}}{\underbrace{(3n+3)!}_{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}}$$

$$= \lim_{n\to\infty} \underbrace{\frac{(n+1)^n}{n^n}}_{\to e} \cdot \frac{\cancel{n+1}}{\underbrace{3n+3}_{3\cancel{(n+1)}}} \cdot \underbrace{\frac{n+1}{3n+2}}_{\to \frac{1}{3}} \cdot \underbrace{\frac{1}{3n+1}}_{\to 0} = 0<1$$

∞

Therefore, $\lim_{n\to\infty} \frac{n^n \cdot n!}{(3n)!} = 0$ because otherwise, if $\lim_{n\to\infty}\frac{n^n \cdot n!}{(3n)!} \neq 0$ then the

Series $\sum \frac{n^n n!}{(3n)!}$ would Diverge by the n^{th} Term Divergence Test,

which would contradict our proof of Convergence above.

55. $$\int \cos(x^2) - 1 + \frac{x^4}{2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} - 1 + \frac{x^4}{2}\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!} - 1 + \frac{x^4}{2}\,dx$$

$$= \int \underbrace{\cancel{1} - \cancel{\frac{x^4}{2!}}}_{\substack{n=0 \quad n=1 \\ \text{cancelled}}} + \underbrace{\frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots}_{n=2 \text{ higher}} \cancel{-1} + \cancel{\frac{x^4}{2}}\,dx$$

$$= \int \sum_{n=2}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx = \sum_{n=2}^{\infty}\frac{(-1)^n x^{4n+1}}{(2n)!\,(4n+1)} + C$$

56. $$\int \sin(x^2) - x^2\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} - x^2\,dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!} - x^2\,dx$$

$$= \int \underbrace{\cancel{x^2}}_{\substack{n=0\\ \text{cancelled}}} - \underbrace{\frac{x^6}{3!}}_{n=1} + \underbrace{\frac{x^{10}}{5!}}_{n=2} - \cdots - \cancel{x^2}\,dx$$

keep

$$= \int \sum_{n=1}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}\,dx = \sum_{n=1}^{\infty}\frac{(-1)^n x^{4n+3}}{(2n+1)!\,(4n+3)} + C$$

57. $\int 1-\cos(x^2)\,dx = \int 1 - \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!}\,dx = \int 1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx$

$$= \int \cancel{1} - \left(\cancel{1} - \frac{x^4}{2!} - \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots\right) dx$$

n=0 cancelled keep n=1 higher

$$= \int - \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{(2n)!}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n+1}}{(2n)!\,(4n+1)} + C$$

58. $\int 1 - x^2 - e^{-x^2}\,dx = \int 1 - x^2 - \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx = \int 1 - x^2 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$

$$= \int \cancel{1} - \cancel{x^2} - \left(\cancel{1} - \cancel{x^2} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right) dx$$

n=0 n=1 cancelled keep n=2 higher

$$= \int - \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \int \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n!}\,dx$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{n!\,(2n+1)} + C$$

59. $\int \arctan(2x) - 2x + \frac{8x^3}{3}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{2n+1} - 2x + \frac{8x^3}{3}\,dx$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} - 2x + \frac{8x^3}{3}\,dx$$

$$= \int \cancel{2x} - \cancel{\frac{\overset{8}{2^3} x^3}{3}} + \frac{2^5 x^5}{5} - \cdots - \cancel{2x} + \cancel{\frac{8x^3}{3}}\,dx$$

n=0 n=1 cancelled keep n=2 higher

$$= \int \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}\,dx = \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+2}}{(2n+1)(2n+2)} + C$$

60. $\arctan x = \int \frac{1}{1+x^2}\,dx = \int \frac{1}{1-(-x^2)}\,dx = \int \sum_{n=0}^{\infty} (-x^2)^n\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx$ P.S.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + \cancel{C}^{\,0}$$

Expand to solve for $+C$.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + C$$

Test $x=0$

$$\cancel{\arctan 0}^{\,0} = \cancel{0-0+0-\dots}^{\,0} + C \Rightarrow C=0$$

Finally,

$$\boxed{\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$$

61. Method 1: Integration

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx = \int \sum_{n=0}^{\infty} (-x)^n\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + \cancel{C}^{\,0}$$

Expand to Solve for $+C$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + C$$

Test $x=0$

$$\cancel{\ln 1}^{\,0} = \cancel{0-0+0-\dots}^{\,0} + C \Rightarrow C=0$$

Finally,

$$\boxed{\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}}$$

Method 2: Definition/Chart Method

$f(x) = \ln(1+x)$	$f(0) = \ln 1 = 0$
$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$	$f'(0) = \frac{1}{1+0} = 1$
$f''(x) = \frac{-1}{(1+x)^2} = -(1+x)^{-2}$	$f''(0) = \frac{-1}{(1+0)^2} = -1$
$f'''(x) = \frac{2}{(1+x)^3} = 2(1+x)^{-3}$	$f'''(0) = \frac{2}{(1+0)^3} = 2$
$f^{(4)}(x) = \frac{-6}{(1+x)^4} = -6(1+x)^{-4}$	$f^{(4)}(0) = \frac{-6}{(1+0)^4} = -6$
$\vdots$	$\vdots$

Maclaurin Series formula

$$\cancelto{0}{f(0)} + \cancelto{1}{f'(0)}x + \frac{\cancelto{-1}{f''(0)}}{2!}x^2 + \frac{\cancelto{2}{f'''(0)}}{3!}x^3 + \frac{\cancelto{-6}{f^{(4)}(0)}}{4!}x^4 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{\cancel{2}x^3}{3\cdot\cancel{2!}} - \frac{\cancel{6}x^4}{4\cdot\cancel{3!}} + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$$

Match!

62. Method 1: Definition / Chart Method

$f(x) = \cos x \qquad f(0) = \cos 0 = 1$

$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \qquad f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \qquad f'''(0) = \sin 0 = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = \cos 0 = 1$

$\vdots \qquad \vdots$

Maclaurin Series formula

$$\cancelto{1}{f(0)} + \cancelto{0}{f'(0)}x + \frac{\cancelto{-1}{f''(0)}}{2!}x^2 + \frac{\cancelto{0}{f'''(0)}}{3!}x^3 + \frac{\cancelto{1}{f^{(4)}(0)}}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

Method 2: Differentiation

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n \cancel{(2n+1)} x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$(2n+1)! = \cancel{(2n+1)}(2n)!$$

OR // Optional Method 3: Integration

$$\cos x = -\int \sin x \, dx = -\int x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots dx$$

$$= \int -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots dx$$

$$\downarrow$$

$$\cos x = -\frac{x^2}{2} + \frac{x^4}{3!\cdot 4} - \frac{x^6}{5!\cdot 6} + \cdots + \cancel{C}^{\,1}$$

Test $x=0$

$$\cancel{\cos 0}^{\,1} = -0 + \cancel{0-0}^{\,0} + \cdots + C \Rightarrow C = 1$$

Finally, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

63. Method 1: Definition | Chart Method

$f(x) = \sin x$ $\qquad f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -\cos 0 = -1$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}(0) = \sin 0 = 0$

$\vdots \qquad\qquad \vdots$

Maclaurin Series formula

$$\cancel{f(0)}^{\,0} + \cancel{f'(0)}^{\,1} x + \frac{\cancel{f''(0)}^{\,0}}{2!} x^2 + \frac{\cancel{f'''(0)}^{\,-1}}{3!} x^3 + \frac{\cancel{f^{(4)}(0)}^{\,0}}{4!} x^4 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Method 2: Differentiation

$$\sin x = \frac{d}{dx}\left(-\cos x\right) = \frac{d}{dx}\left(-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\right)$$

$$= \frac{d}{dx}\left(-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots\right)$$

$$= 0 + \frac{1}{\cancel{2!}} \cdot \cancel{2}x - \frac{1}{\underset{\cancel{4}\cdot 3!}{\cancel{4!}}} \cdot \cancel{4}x^3 + \frac{1}{\underset{\cancel{6}\cdot 5!}{\cancel{6!}}} \cdot \cancel{6}x^5 - \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Method 3: Integration

$$\sin x = \int \cos x \, dx = \int 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \, dx$$

$$= x - \frac{x^3}{2! \cdot 3} + \frac{x^5}{4! \cdot 5} - \frac{x^7}{6! \cdot 7} + \dots + C$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + \cancel{C}^{\,0}$$

Test $x=0$

$$\cancel{\sin 0}^{\,0} = \cancel{0 - 0 + 0 - 0 + \dots}^{\,0} + C \Rightarrow C = 0$$

Finally, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

64. Method 1: Integration

$$\ln(3+x) = \int \frac{1}{3+x} dx = \int \frac{1}{3\left(1+\frac{x}{3}\right)} dx = \int \frac{1}{3} \cdot \frac{1}{1-\left(-\frac{x}{3}\right)} dx$$

$$= \int \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n dx = \int \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)} + C$$

Expand to solve for $+C$

$$\ln(3+x) = \underbrace{\frac{x}{3 \cdot 1}}_{n=0} - \underbrace{\frac{x^2}{3^2 \cdot 2}}_{n=1} + \underbrace{\frac{x^3}{3^3 \cdot 3}}_{n=2} - \frac{x^4}{3^4 \cdot 4} + \dots + C$$

Test $x=0$

$$\ln 3 = \cancel{0 - 0 + 0 - \dots}^{\,0} + C \Rightarrow C = \ln 3$$

Finally, $\ln(3+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)} + \ln 3$

Method 2: Substitution

$$\ln(3+x) = \ln\left(3\left(1+\frac{x}{3}\right)\right) \overset{\text{Log Algebra}}{=} \ln 3 + \ln\left(1+\frac{x}{3}\right)$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{3}\right)^{n+1}}{n+1}$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}(n+1)}$$

Match!

OR

Method 3: Definition / Chart Method

$f(x) = \ln(3+x)$	$f(0) = \ln 3$
$f'(x) = \frac{1}{3+x}$	$f'(0) = \frac{1}{3}$
$f''(x) = \frac{-1}{(3+x)^2}$	$f''(0) = -\frac{1}{3^2}$
$f'''(x) = \frac{2}{(3+x)^3}$	$f'''(0) = \frac{2}{3^3}$
$f^{(4)}(x) = \frac{-6}{(3+x)^4}$	$f^{(4)}(0) = -\frac{6}{3^4}$
$\vdots$	$\vdots$

Maclaurin Series formula

$$\overset{\ln 3}{\cancel{f(0)}} + \overset{\frac{1}{3}}{\cancel{f'(0)}}\,x + \frac{\overset{-\frac{1}{3^2}}{\cancel{f''(0)}}}{2!}x^2 + \frac{\overset{\frac{2}{3^3}}{\cancel{f'''(0)}}}{3!}x^3 + \frac{\overset{-\frac{6}{3^4}}{\cancel{f^{(4)}(0)}}}{4!}x^4 + \cdots$$

$$= \ln 3 + \frac{1}{3}x - \frac{1}{2!\,3^2}x^2 + \frac{\cancel{2}}{\underset{3}{\cancel{3!}}\cdot 3^3}x^3 - \frac{\overset{3\cdot 2}{\cancel{6}}}{\underset{4\cdot\cancel{3\cdot 2}}{4!}\,3^4}x^4 + \cdots$$

$$= \ln 3 + \frac{x}{3} - \frac{x^2}{3^2\cdot 2} + \frac{x^3}{3^3\cdot 3} - \frac{x^4}{3^4\cdot 4} + \cdots + C$$

$$= \ln 3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{3^{n+1}\cdot(n+1)}$$