Review Packet for Exam #3

Math 121-D. Benedetto

Interval of Convergence: Find the **interval** and **radius of convergence** for each of the following power series. Analyze convergence at the endpoints carefully, with full justification.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+3)^n}{n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{n^2 4^n}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n (x+3)^n}{(n+1)^3 n!}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n2^n}{n+5}(x+1)^n$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+2)!(x-5)^n}{10^n}$

6. $\displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{n}(2x-1)^n}{4^n}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^n}$

9. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2} x^n$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(3n)!} x^n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^3}(x-1)^n$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^{\frac{1}{2}}}$

13. $\displaystyle\sum_{n=1}^{\infty} nx^n$

14. Challenge: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Estimates: Use a Power Series Representation for each of the following functions to **estimate** each one within the given error.

15. Estimate $\cos(1)$ with error less than $\frac{1}{100}$
16. Estimate $e^{-\frac{1}{3}}$ with error less than $\frac{1}{100}$
17. Estimate $\arctan 1$ with error less than $.20$
18. Estimate $\frac{1}{e}$ with error less than $\frac{1}{10}$
19. Estimate $\sin(1)$ with error less than $\frac{1}{100}$
20. Estimate $\frac{1}{\sqrt{e}}$ with error less than $\frac{1}{100}$
21. Estimate $\sin\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$
22. Estimate $\arctan\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$
23. Estimate $\ln 2$ with error less than $\frac{1}{5}$
24. Estimate $\cos\left(\frac{1}{2}\right)$ with error less than $\frac{1}{100}$
25. Estimate $\ln\left(\frac{3}{2}\right)$ with error less than $\frac{1}{10}$

MacLaurin Series: Find the MacLaurin Series for each of the following functions, as well as the corresponding radius of convergence.

26. $f(x) = x^2 e^{-3x^4}$
27. $f(x) = \dfrac{1 - e^{-x}}{x}$
28. $x^4 \ln(1 + x^3)$
29. $\cosh x$
30. $\sinh x$
31. $f(x) = \dfrac{x^6}{1 + 7x}$
32. $f(x) = x \arctan(2x)$

Power Series Representations of Functions: Use a Power Series Representation for each of the following functions to compute the given integral. Estimate each one within the given error.

33. Estimate $\displaystyle\int_0^1 x^2\cos(x^3)\,dx$ with error less than $\dfrac{1}{50}$.

34. Estimate $\displaystyle\int_0^{\frac{1}{2}} x\arctan x\,dx$ with error less than 0.01.

35. Estimate $\displaystyle\int_0^1 \sin(x^2)\,dx$ with error less than 0.1.

36. Estimate $\displaystyle\int_0^{\frac{1}{2}} e^{-x^3}\,dx$ with error less than 0.01.

Sums: Find the **sum** for each of the following series. (hint: you are allowed to pull an x out of these sums in n. For the harder ones, can you recognize the series as a derivative or integral of another function's power series representation?) Your answer may include x.

37. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 2^{n+2}}{3^n}$

38. $1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dots$

39. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}$

40. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n 49^n\pi^{2n}}{4^n(2n+1)!}$

41. $\displaystyle\sum_{n=0}^{\infty}\frac{(-9)^n\pi^{2n+1}}{4^n(2n)!}$

42. $\displaystyle\sum_{n=0}^{\infty}\frac{(-\pi^2)^n}{36^n(2n)!}$

43. $\displaystyle\sum_{n=0}^{\infty}\frac{x^{7n+1}}{n!}$

44. $1-\dfrac{1}{2}+\dfrac{1}{2^2\,2!}-\dfrac{1}{2^3\,3!}+\dfrac{1}{2^4\,4!}+\dots$

45. $-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\dfrac{1}{9}-\dots$

46. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}(n+1)}$

Volumes of Revolution: Find the following volumes requested. Make sure to draw one of the Approximating Disks, Washers, or Cylindrical Shells in your diagram. This will help you set up your integral(s).

47. Consider the region bounded by $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$, and the x-axis. Rotate the region about the y-axis and find the volume of the resulting solid using the Cylindrical Shell Method.

48. Consider the same region bounded by $y = \cos x$, $x = 0$, $x = \frac{\pi}{2}$, and the x-axis. Now rotate the region about the x-axis and find the volume of the resulting solid using the Disk Method.

49. Consider the region bounded by $y = \ln x$, $x = 1$, $x = e$, and the x-axis. Rotate the region about the y-axis and find the volume of the resulting solid using the Cylindrical Shell Method.

50. Consider the same region bounded by $y = \ln x$, $x = 1$, $x = e$, and the x-axis. Rotate the region about the x-axis and find the volume of the resulting solid using the Disks Method. What would be the set-up for using the Shell Method? Try and set it up at least...

51. Consider the region bounded by $y = e^x$, $x = 1$, $x = 2$, and the x-axis. Rotate the region about the y-axis and find the volume of the resulting solid using the Cylindrical Shell Method.

52. Consider the region bounded by $y = e^x$, $x = 0$, $x = 2$, and the x-axis. Rotate the region about the line $y = -1$ and find the volume of the resulting solid using the Washer Method. Why is the Washer Method more helpful here than say the Cylindrical Shells Method? Can you set-up the integral using Shells?

53. Consider the same region bounded by $y = e^x$, $x = 0$, $x = 2$, and the x-axis. Rotate the region about the line $x = 4$ and find the volume of the resulting solid using the Cylindrical Shell Method.

54. Consider the same region bounded by $y = e^x$, $x = 0$, $x = 2$, and the x-axis. Rotate the region about the line $x = -1$ and find the volume of the resulting solid. Which method would be helpful?

55. Consider the region bounded by $y = e^x$, $y = x$, $x = 0$ and $x = \ln 3$. Rotate the region about the y-axis and find the volume of the resulting solid. Which method would be helpful?

56. Consider the same region bounded by $y = e^x$, $y = x$, $x = 0$, $x = \ln 3$, and the x-axis. Rotate the region about the x-axis and find the volume of the resulting solid. Which method would be helpful?

57. Consider the region bounded by $y = \sqrt{x-1}$, $x = 5$, $x = 10$ and the x-axis. Rotate the region about the line $x = -2$ and find the volume of the resulting solid. Which method would be helpful?

Parametric Equations: Answer each of the following questions, related to the given parametric equations.

58. Let the curve represented by the parametric equations $x = t + \frac{1}{t}$ and $y = 2\ln t$ for $1 \le t \le 3$.

    (a) Find the equation of the tangent line to the curve at the point $(\frac{5}{2}, 2\ln 2)$.

    (b) Find the arclength of this parametric curve for $1 \le t \le 3$.

59. Let the curve represented by the parametric equations $x = \tan t - t$ and $y = \ln(\cos t)$ for $0 \le t \le \frac{\pi}{3}$.

    (a) Find $\dfrac{dy}{dx}$ for the curve when $t = \frac{\pi}{6}$.

    (b) Find the arclength of this parametric curve for $0 \le t \le \frac{\pi}{3}$. (hint: $\sec^2 t - 1 = \tan^2 t$)

60. Let the curve represented by the parametric equations $x = t - e^t$ and $y = 1 - 4e^{\frac{t}{2}}$ for $0 \le t \le \ln 5$.

    (a) Find $\dfrac{dy}{dx}$ for the curve when $t = \ln 4$.

    (b) Find the arclength of this parametric curve for $0 \le t \le \ln 5$.

    (c) Set-up (but do not evaluate) the definite integral representing the surface area of the figure obtained by revolving this curve around the x-axis for $0 \le t \le \ln 5$.

61. Let the curve represented by the parametric equations $x = e^t \cos t$ and $y = e^t \sin t$

    for $0 \le t \le \ln \pi$.

    (a) Find the arclength of this parametric curve for $0 \le t \le \ln \pi$.

62. Let the curve represented by the parametric equations $x = 3t^2$ and $y = 2t^3$ for $0 \le t \le \ln 3$.

    (a) Find the equation of the tangent line to the curve at the point $(3, 2)$.

    (b) Find the arclength of this parametric curve for $0 \le t \le 1$.

    (c) Find the surface area obtained by rotating this curve about the y-axis for $0 \le t \le 1$.

63. Let the curve represented by the parametric equations $x = \sin^3 t$ and $y = \cos^3 t$ from $t = 0$ to $t = \frac{\pi}{2}$.

    (a) Find the equation of the tangent line to the curve at the point$(\frac{3\sqrt{3}}{8}, \frac{1}{8})$.

    (b) Find the arclength of this parametric curve for $0 \le t \le \dfrac{\pi}{2}$.

    (c) Find the surface area obtained by rotating this curve about the x-axis for $0 \le t \le \dfrac{\pi}{2}$.

64. Let the curve represented by the parametric equations $x = 3 - 2t$ and $y = e^t + e^{-t}$.

    (a) Find the arclength of this parametric curve for $0 \le t \le 1$.

    (b) Find the surface area obtained by rotating this curve about the x-axis for $0 \le t \le 1$.

    (c) Set-up (but do not evaluate) the definite integral representing the surface area of the figure obtained by revolving this curve around the y-axis for $0 \le t \le 1$.

Limits: Compute each of the following limits in two ways: first using L'H Rule and second using series.

65. $\displaystyle\lim_{x\to 0} \frac{\sin(3x) - 3x}{x - \arctan x}$

66. $\displaystyle\lim_{x\to 0} \frac{xe^x - \arctan x}{\ln(1+3x) - 3x}$

Sequence Limits:

67. Use Series to show that $\displaystyle\lim_{n\to\infty} \frac{6^n}{n!} = 0$

68. Use Series to show that $\displaystyle\lim_{n\to\infty} \frac{n^n\ n!}{(3n)!} = 0$

Integrals:

69. Use Series to compute $\displaystyle\int \cos\left(x^2\right) - 1 + \frac{x^4}{2}\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=2}^{\infty}$.

70. Use Series to compute $\displaystyle\int \sin\left(x^2\right) - x^2\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=1}^{\infty}$.

71. Use Series to compute $\displaystyle\int 1 - \cos\left(x^2\right)\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=1}^{\infty}$.

72. Use Series to compute $\displaystyle\int 1 - x^2 - e^{-x^2}\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=2}^{\infty}$.

73. Use Series to compute $\displaystyle\int \arctan(2x) - 2x + \frac{8x^3}{3}\ dx$. Your answer should be in sigma notation $\displaystyle\sum_{n=2}^{\infty}$.

Derivative Values: Hint: Do not compute out the following derivatives manually.

Hint: Write out the definition of the MacLaurin Series for any $f(x)$.

74. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = x^5 \sin(x^3)}$. State the Radius of Convergence.

    (b) Use this series to determine the **eighth** and **ninth** derivatives of $f(x) = x^5 \sin(x^3)$ at $x = 0$. **Simplify** here.

75. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = xe^{-x^7}}$. State the Radius of Convergence.

    (b) Use this series to determine the **twenty first** and **twenty second** derivatives of $f(x) = xe^{-x^7}$ at $x = 0$. Do not simplify here.

76. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = x^5 \ln(1 + 3x)}$. State the Radius of Convergence.

    (b) Use this series to determine the **seventh** and **ninth** derivatives of $f(x) = x^5 \ln(1 + 3x)$ at $x = 0$. Do not simplify here.

77. (a) Write the MacLaurin Series for $\boldsymbol{f(x) = x \arctan\left(x^2\right)}$. State the Radius of Convergence.

    (b) Use this series to determine the **seventh** and **eighth** derivatives of $f(x) = x \arctan\left(x^2\right)$ at $x = 0$. **Simplify** here.

78. (a) Find the MacLaurin Series for $\cosh x$.

    (b) Demonstrate a **second, different** method/approach from part (a) above, to compute the MacLaurin Series for the same function, $f(x) = \cosh x$.

    (c) Demonstrate a **third, different** method/approach from parts (a) and (b) above, to compute the MacLaurin Series for the same function, $f(x) = \cosh x$.

    (d) Find the MacLaurin Series for $f(x) = \cosh\left(3x^2\right)$.

    (e) Use this series to determine the **seventh** and **eighth** derivatives of $f(x) = \cosh\left(3x^2\right)$ at $x = 0$. Do not simplify here.

Challenging Sums: Find the sum for each of the following convergent series.

79. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{4^n}$

80. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{4^n}$

81. $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(\ln 3)^n}{n!}$

82. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$

83. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1}$